A COMPENDIOUS TREATISE ON THE ELEMENTS OF PLANE TRIGONOMETRY

WITH

THE METHOD OF CONSTRUCTING

TRIGONOMETRICAL TABLES.

BY THE

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SECOND EDITION

LONDON:

PRINTED BY R. WATTS;

FOR T. CADELL AND W. DAVIES, STRAND, LONDON;

DEIGHTONS, NICHOLSONS, AND BARRETT, CAMBRIDGE;

AND PARKERS, OXFORD.

1818

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PLANE TRIGONOMETRY

CHAP. I.

INTRODUCTION.

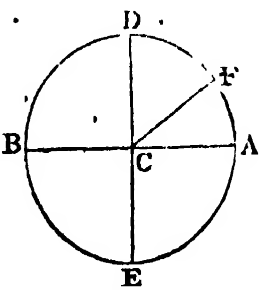
I.

DEFINITIONS.

1. PLANE Trigonometry is that branch of Mathematics, by which we investigate the relation which obtains between the sides and angles of plane triangles.

2. In order to make this investigation, it is necessary to obtain a proper representation for the *measure of an angle*.

Describe the circle $ADBE$, and draw two diameters AB, DE, at right angles to each other, which will divide the circumference into four equal parts, AD, DB, BE, EA, each of which is called a *quadrant*. Draw any line CF from the centre to the circumference; then (Euc. 6. 33.) the angles ACF, ACD, are to each other as the arcs AF, AD; so that if the magnitude of the angle ACF be represented by the arc AF, the



magnitude of the angle ACD will be represented by the arc AD; and so of any other angles; i.e. *the magnitude of an angle is measured by the arc which subtends it in a circle described with a given radius.*

3. For the purpose of exhibiting *arithmetically* the magnitude of angles, the whole circumference of the circle is supposed to be divided into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each minute into 60 equal parts, called *seconds;* &c. &c. And since arcs are the *measures* of angles, every angle may be said to be an angle of such number of degrees, minutes, and seconds, as the arc subtending it contains. Thus, if the arc AF contains 38 degrees 14 minutes 25 seconds, the angle ACF (adopting the common notation of °, ′, ″, &c for *degrees, minutes, seconds,* &c.) is said to be an angle of 38° 14′ 25″. The *quadrants* AD, DB, BE, EA evidently contain 90° each.

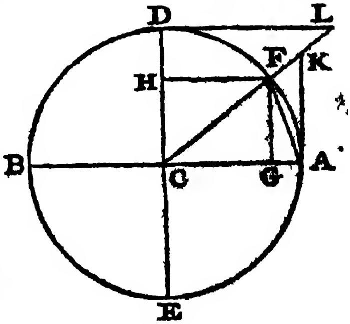
4. The difference between any angle ACF and a right angle or 90°, is called the *complement* of that angle. Thus, if ACF is an angle of 37° 5′ 2″, its *complement* FCD will be an angle of 52° 54′ 58″.

5. The *supplement* of an angle is the difference between it and 180°. Thus. if the angle ACF is 40° 25′ 35″, its *supplement* FCB will be 139° 34′ 25″.*

6 The

* Since the three angles of every triangle are equal to *two right angles*, or to 180°, it is evident that in a *right-angled* triangle the two *acute* angles must be together equal to one right angle, or 90°, the acute angles must therefore be the *complements* the

6. The straight line *AF*, drawn from one extremity of the arc to the other, is called the *chord* of the arc *AF*.



7. *FG*, a line drawn from *one* extremity of the arc *AF* perpendicular upon the diameter *(AB)* passing through the *other* extremity, is called the *sine* of the angle *ACF*.

8. *AG*,

---

the one of the other; and in an *oblique-angled* triangle, the *third* angle must be the *supplement* of the sum of the other two angles.

In the French division of the circle, the whole circumference is supposed to be divided into 400 equal parts, called *degrees*, each degree into 100 *minutes*; each minute into 100 *seconds*, &c. &c. so that, according to this scale, 47 degrees 15 minutes 17 seconds may be expressed by $47° 15' 17''$, or by $47°.1517$, where the decimal .1517 is the fractional part of a degree corresponding to the 15 minutes and 17 seconds.

The degrees, minutes, &c. of the French scale are converted into degrees, minutes, &c of the English scale by a very simple Arithmetical process. For since the quadrant, according to the *former* scale, consists of 100°, and, according to the *latter*, of 90°, the number of degrees in any given arc or angle, according to the *English* scale, must be $\frac{9}{10}$ths of that number on the *French* scale. From the degrees therefore of the French scale, we must *subtract* $\frac{1}{10}$th, and it will give the number of *degrees* upon the English scale; then multiplying the *decimal* part of the resulting quantity by 60, it will give the number of *minutes*; and

8. *AG*, that part of the diameter which is intercepted between the extremity of the arc *AF*, and the sine *FG*, is called the *versed sine* of the angle *ACF*.

9. If a line be drawn touching the circle in *A*, and the radius *CF* be produced to meet it in *K*, then *AK* is called the *tangent*, and *CK* the *secant* of the angle *ACF*.

10. If

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and the decimal part of the *minutes* by 60, it will give the number of *seconds*; &c. &c. as in the following examples.

Subtract $\frac{1}{10}$th } 76° Fr. sc. / 7.6	24°. 15 French / 2 415 = $\frac{1}{10}$th	47°. 1517 French / 4 .71517 = $\frac{1}{10}$th
68 . 4	21 735	42 . 43653
60	60	60
24 . 0	44 . 100	26 . 19180
	60	60
	6 000	11 . 50800
∴ 76° *French* = 68° 24 *English*	∴ 24° 15′ *French* = 21° 44′ 6″ *English*	∴ 47° 15′ 17″ *Fr* = 42° 26′ 11″ *Eng*

Since 90° *English* make 100° *French*, to convert English degrees, minutes, &c. into French ones of the same value, we must reduce the former into degrees and decimals of a degree, and then *add* $\frac{1}{9}$th. For example, let it be required to reduce 23° 27′ 58″ *English*, to *French* ones of the same value.

$$\left.\begin{array}{l} 27' = \frac{27}{60} \text{ of a degree } = .4500 \\ 58'' = \frac{58}{3600} \quad . \quad . \quad . = 0161 \end{array}\right\}$$

Hence 23° 27′ 58″ = 23.4661.

Add $\frac{1}{9}$th = 2.6074.

Then 26.0735, or 26° 7′ 35″, are the [number of *French*.

10. If a line be drawn touching the circle in D, and CF be produced to meet it in L, and FH be let fall perpendicular upon the diameter (DE), then FH, DH, DL, and CL become respectively the sine, versed sine, tangent, and secant of the angle FCD, which is the *complement* of the angle ACF, and are therefore called the *cosine*, *co-versed sine*, *cotangent*, and *cosecant* of the angle ACF.

11. Since CG is equal to FH, it is equal to the *cosine* of the arc AF; hence the *cosine* of any arc is that part of the radius of the circle which is intercepted between the *centre of the circle* and the *extremity of the sine* of that arc.

II.

On the general relation which the sine, cosine, versed sine, tangent, secant, cotangent, and cosecant, of any arc or angle bear to each other, and to the radius of the circle.

In this investigation, the following *abbreviations* are used; viz.

sin.	for sine.	*sec.*	for secant.
cos.	... cosine.	*cotan.*	... cotangent.
v. sin.	... versed sine.	*cosec.*	... cosecant.
tan.	... tangent.	*diam.*	... diameter.

In the *right-angled* triangle CFG, we have (Euc. 47. 1.)

12. $$FG = \sqrt{CF^2 - CG^2},$$

$$\text{i.e. sine} = \sqrt{\text{rad.}^2 - \text{cosin.}^2}$$

And, *vice versa*,

13. $$CG = \sqrt{CF^2 - FG^2},$$

$$\text{i.e. cosine} = \sqrt{\text{rad.}^2 - \text{sin.}^2}$$

14. AG

14. $AG = AC - CG$,

i. e. versed sine = rad. − cos.

15. By similar triangles ACK, GCF,

$AK : AC :: FG : CG$,

i. e. tangent : radius :: sine : cosine, or tan. $= \frac{\text{rad.} \times \text{sin.}}{\text{cos.}}$

16. By similar triangles ACK, DCL,

$AK : AC :: CD : DL$,

i. e. tangent : radius :: radius : cotan. $= \frac{\text{rad.}^2}{\text{tan.}}$

17. By similar triangles ACK, GCF,

$CK : CA :: CF : CG$,

i. e. secant : radius :: radius : cosine, or sec. $= \frac{\text{rad.}^2}{\text{cos.}}$

18. In the right-angled triangle CAK, we have

$$CK = \sqrt{CA^2 + AK^2},$$

i. e. secant $= \sqrt{\text{rad.}^2 + \text{tan.}^2}$

And, *vice versa*,

$$AK = \sqrt{CK^2 - AC^2},$$

i. e. tangent $= \sqrt{\text{sec.}^2 - \text{rad.}^2}$

19. By similar triangles DCL, GCF,

$CL : CD :: CF : FG$,

i. e. cosecant : radius :: radius : sine, or cosec. $= \frac{\text{rad.}^2}{\text{sin.}}$

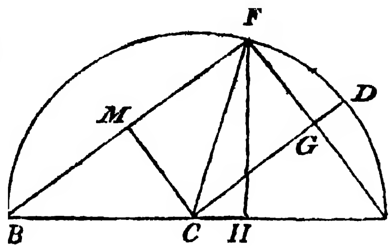
III.

A few Properties of Arcs and Angles demonstrated geometrically.

PROPERTY 1.

20. *The chord of any arc is a mean proportional between the versed sine of that arc and the diameter of the circle.*

AF is the *chord*, and *AH* is the *versed sine* of the arc *AF*; join *FB*, then the angle *AFB* in a semicircle is a right angle; ∴ since *FH* is perpendicular to *AB*, we have, (Eucl. 6. 8.)



$$AH \; : \; AF \qquad AF \qquad AB,$$

i. e. V. sin. : chord   chord   diam.

PROP. 2.

21. *The chord of an arc is double the sine of half that arc.*

Draw *CG* at right angles to *AF*, and produce it to *D*; then (Eucl. 3. 3.) *CG* bisects the *chord AF*; and (Eucl. 3. 30.) it also bisects the *arc AF*. Hence,

Chord $AF = 2FG$, and *arc* $AF = 2FD$, or $FD = \frac{1}{2}AF$.

Now FG = sine of arc FD = sine of $\frac{1}{2}$ arc AF;

∴ *Chord* AF $(=2FG)$ = *twice sine of* $\frac{1}{2}$ *arc* AF.

And, *vice versa*;

Since $FG = \frac{1}{2}$ chord of arc AF $(= \frac{1}{2}$ chord $2FD)$, we have *sine of an arc* = $\frac{1}{2}$ *chord of double the arc.*

PROP.

Prop. 3.

22. *As radius : cosine of any arc :: twice the sine of that arc : the sine of double the arc.*

For CG = cosine of arc FD,
AF (= $2FG$) = twice the sine of arc FD,
FH (= sine of AF) = sine of double the arc FD.

Now the *right-angled* triangles ACG, AFH, have a common angle at A, they are consequently *similar*; hence $AC : CG :: AF : FH$,
i. e. *radius : cos. of arc FD :: twice the sine of arc FD : sine of double the arc.*

Prop. 4.

23. *Half the chord of the supplement of any arc is equal to the cosine of half that arc.*

Draw CM at right angles to BF; then since CG is parallel to BF, and CM parallel to AF, the figure $FGCM$ is a *parallelogram*; $\therefore MF = CG$; but $MF = (\frac{1}{2}FB =)$ $\frac{1}{2}$ chord of the *supplemental* arc FB, and CG = cosine of FD, which is $\frac{1}{2}$ the arc AF;
Hence, *Half the chord of the supplement of the arc AF is equal to the cosine of half the arc AF.*

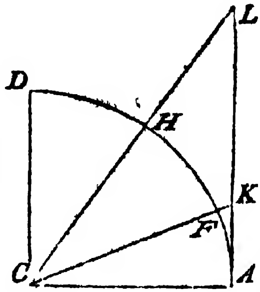
Prop. 5.

24. *Tangent + secant of any arc is equal to the cotangent of half the complement of that arc.* (Fig. in p. 9.)

Let AD be the quadrant of a circle, AF any arc, whose *tangent* is AK, *secant* CK, and *complement* the arc FD.

Bisect

Bisect FD in H, join CH, and produce CH and AK to meet in L; then AL is the *tangent* of the arc AH, and consequently the *cotangent* of the arc HD, which is *half the complement* of the arc AF.



Now, since AL is parallel to CD, the angle DCH is equal to the angle CLK; but DCH is equal to HCK, $\therefore$ CLK is equal to HCK, and consequently $KL = CK$.

Now $AK + KL = AL$;

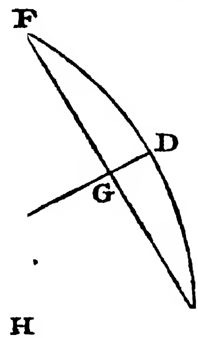
$\therefore \quad AK + CK = AL$, i.e.

tang. + *secant* = *cotang. of half complement of arc* AF.

Prop. 6.

25. *The chord of* 60° *is equal to the radius of the circle.*

Let AF be an arc of 60°, then angle ACF of the triangle ACF is 60°; and since the three angles of the triangle are equal to 180°, the two remaining angles CAF, CFA, must be equal to 120°; but $CA = CF$, $\therefore$ $\angle CAF = CFA$, and each of them are 60°; hence the triangle CAF is *equiangular*, and consequently *equilateral*; wherefore chord AF ($= AC$ or CF) = rad.



PROP. 7.

26. *The sine of 30° is equal to half the radius.*

By Prop. 2. the sine of an arc is half the chord of double the arc; if therefore AF is 60°, FD will be 30°, and its *sine* $FG=\frac{1}{2}AF=$ (by Prop. 6.) $\frac{1}{2}$ the radius.

PROP. 8.

27. *The versed sine and cosine of 60° are each equal to half the radius.*

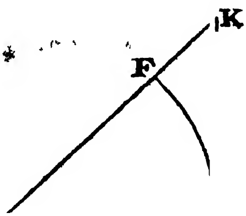
For since the triangle AFC is *equilateral*, the sine FH *bisects* the base (or radius) AC. Hence,

$AH=$ *versed sine* of 60° = half the radius.
$CH=$ *cosine* of 60° = half the radius.

PROP. 9.

28. *The tangent of 45° is equal to the radius.*

Let arc $AF=45°$, then the angle $ACK=45°$; and since $\angle CAK=90°$, the remaining angle AKC must be 45°; hence $\angle ACK=$ $=\angle AKC$, $\therefore$ the tangent AK $=AC=$ radius.



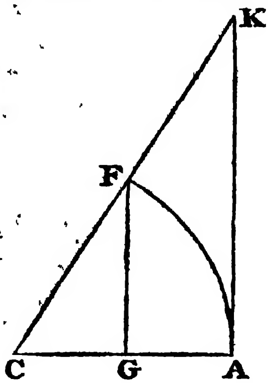
PROP. 10.

29. *The secant of 60° is equal to the diameter of the circle.*

Let arc $AF=60°$, draw the *tangent* AK, and *secant* CK; then, by Prop. 8. $CG=GA$; and since FG is parallel to AK,

$$CF : FK :: CG : GA.$$

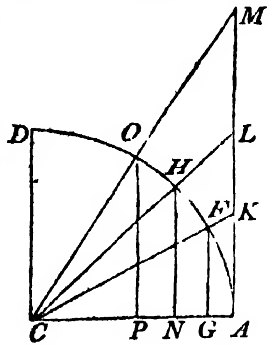
But $CG=GA$, $\therefore$ $CF=FK$; hence $CK=2CF=2$ rad. $=$ diam.



IV.

The sine, cosine, tangent, and secant, of 30°, 45°, and 60°, exhibited arithmetically.

Let AD be a quadrant of a circle, and AF, AH, AO, arcs of 30°, 45°, and 60°, respectively. In tracing the value of the sine, tangent, and secant, from A to D, it is evident that at A, when the arc $=0$, the *sine* and *tangent* are each equal to 0, but that the *secant* is equal to *radius*. In proceeding from A to D, these lines keep continually increasing, and in such manner, that at D the *sine* of AD or 90° becomes equal to the *radius* CD; the *tangent* and *secant* of AD (being formed by the intersection of two lines, one drawn touching the circle in A, the other at right angles to AC in the point C, and consequently *parallel*) become both *indefinitely great*. At A the *cosine* $=CA=$ *radius*; and as the arc *increases* the cosine *decreases*, so that when the arc becomes 90°, the *cosine* is equal to 0. Our object at present is, to find *arithmetically* the value of the sine, cosine, tangent, and secant, at the *intermediate* points F, H, O, on supposition that the radius is equal to *unity*.



30. *Value*

30. *Value of Sines FG, HN, OP.*

FG = sin. of 30° = (by Art. 25.) $\frac{1}{2}$ rad. = (if rad. = 1) $\frac{1}{2}$ = .5000000.

Since $\angle HCN = 45^\circ$, CHN also $= 45^\circ$, $\therefore CN = HN$;
hence, $CH^2 = (CN^2 + HN^2 =)\ 2\,HN^2$, or
$HN^2 = \frac{CH^2}{2}$; $\therefore HN = \sin.45^\circ = \frac{CH}{\sqrt{2}} = \frac{1}{\sqrt{2}} = .7071068$,*

$OP = \sin.60^\circ = \sqrt{CO^2 - CP^2}$ = (for $CP = \frac{1}{2}$, by Art. 27.)
$\sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = .8660254.$†

31. *Value of Cosines CG, CN, CP.*

CG = cosine of 30° = sine of 60° = $\frac{\sqrt{3}}{2}$ = .8660254.

CN = cosine of 45° = HN = $\frac{1}{\sqrt{2}}$ = .7071068.

CP = cosine of 60° = sine of 30° = $\frac{1}{2}$ = .5000000.

32. *Value of Tangents AK, AL, AM.*

By Art. 15. $\tan = \frac{\text{rad.} \times \sin.}{\cos.}$ = (if rad. = 1) $\frac{\sin.}{\cos.}$

Hence $AK = \tan.30^\circ = \frac{\sin.30^\circ}{\cos.30^\circ} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = .5773503.$

$AL = \tan.45^\circ = \frac{\sin.45^\circ}{\cos.45^\circ} = \frac{HN}{CN}$ = (as $HN = CN$) 1.0000000.

$AM = \tan.60^\circ = \frac{\sin.60^\circ}{\cos.60^\circ} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3} = 1.7320508.$

33. *Value*

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* For $\sqrt{2} = 1.4142136$. † For $\sqrt{3} = 1.7320508$.

33. *Value of Secants CK, CL, CM.*

By Art. 17. sec. $=\frac{\text{rad.}^2}{\text{cos.}}=$ (if rad. $=1$) $\frac{1}{\text{cosine}}$.

Hence $CK=\text{sec}.30^\circ=\frac{1}{\cos.30^\circ}=1\times\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}=1.1547005.$

.... $CL=\text{sec}.45^\circ=\frac{1}{\cos.45^\circ}=1\times\frac{\sqrt{2}}{1}=\sqrt{2}=1.4142136.$

.... $CM=\text{sec}.60^\circ=\frac{1}{\cos.60^\circ}=1\times\frac{2}{1}=2=2.0000000.$

V.

34. *On finding the sines of various arcs, by means of the expression for finding the sine of half an arc.*

By Art. 20, we have

Ver. sine of an arc : chord :: chord : diameter.

But the *chord* of any arc is equal to *twice the sine of* $\frac{1}{2}$ *that arc*, and the *diameter* is equal to *twice the radius*. Hence, by substitution,

Ver. sin. of an arc : $2\times$ sin. of $\frac{1}{2}$ arc :: $2\times$ sin. of $\frac{1}{2}$ arc : $2\times$ radius.

$$\therefore 4\times\overline{\text{sin. of }\tfrac{1}{2}\text{ arc}}|^2=2\times\text{ver. sin.}\times\text{rad.}$$

$$\text{or}\quad\overline{\text{sin. of }\tfrac{1}{2}\text{ arc}}|^2=\frac{\text{v. sin.}\times\text{rad.}}{2}$$

$$\text{and, sin. of }\tfrac{1}{2}\text{ arc}=\sqrt{\frac{\text{v. sin.}\times\text{rad.}}{2}}$$

If therefore the radius $=1$, the *sine of* $\frac{1}{2}$ *an arc* is equal to the *square root of* $\frac{1}{2}$ *the versed sine of that arc;* and since the *versed sine* of an arc is equal to *rad.* $-$ *cos.* (Art. 14.), we have *sine of* $\frac{1}{2}$ *arc* $=\sqrt{\frac{1-\cos.}{2}}$.

Now

Now

$$\cos.30^\circ = .8660254, \therefore \sin.15^\circ = \sqrt{\frac{1-.8660254}{2}} = .2588190,$$

$$\text{and } \cos.15^\circ = \sqrt{1-\overline{\sin.}|^2} = .9659258.$$

$$\text{Hence, sine } 7^\circ 30' = \sqrt{\frac{1-.9659258}{2}} = .1305262.$$

$$\text{cosine } 7^\circ 30' = \text{\&c.}$$

$$\text{sine } 3^\circ 45' = \text{\&c.}$$

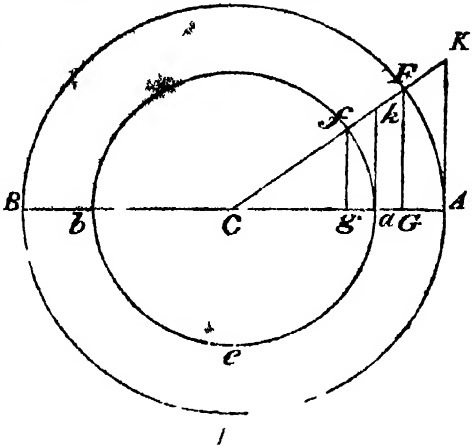
And thus, by *halving* each preceding angle, we might find the value of the sines and cosines of a series of angles continually decreasing without limit. From the cosine of 45° we might also find the sine and cosine of *another* series of angles, 22° 30; 11° 15′; &c. &c. decreasing in the same manner. Having the *sine* and *cosine* of an angle, its *tangent*, *secant*, &c. may be found from the expressions in Sect II., viz tan. $= \frac{\text{rad. sin.}}{\text{cos.}}$; sec. $= \frac{\text{rad.}^2}{\text{cos.}}$; cotan. $= \frac{\text{rad.}^2}{\text{tan.}}$; and cosec. $= \frac{\text{rad}}{\text{sin}}$

In this manner, from the sine and cosine of 45° and 30°, we might find the sine, cosine, tangent, secant, &c. of a vast variety of angles less than 22° 30′. But the method of constructing arithmetically a complete table of sines, cosines, tangents, &c. for every degree and minute of the quadrant, will form the subject of the Third Chapter.

VI.

On the relation of the sine, tangent, secant, &c. of the same angle in different circles.

35. Let $AFBE$, $afbe$, be two circles whose radii are AC, aC; let an angle be formed at C, subtending the arcs AF, af; draw the *sines* FG, fg; the *tangents* AK, ak; the *secants* CK, Ck; &c. &c.



Now it is evident that

the $\angle ACF$: 4 right $\angle$s :: AF : circumference $AFBE$,
and $\angle aCf$: 4 right $\angle$s :: af : circumference $afbe$.

$$\text{Hence } \angle ACF = 4 \text{ right } \angle\text{s} \times \frac{FA}{AFBE}.$$

$$\angle aCf = 4 \text{ right } \angle\text{s} \times \frac{af}{afbe}.$$

$$\text{But } \angle ACF \text{ is the same with } aCf, \therefore \frac{AF}{AFBE} = \frac{af}{afbe},$$

consequently $AF : af :: AFBE : afbe$,
$:: AC : aC$, since *circumference* of circles are to each other as their *radii*.

Hence it appears, that the *measures* of the same angle in different circles are to each other as the radii of those circles;

circles; and so it is with respect to the *sines*, *tangents*, *secants*, &c. of that angle; for by similar $\triangle^s$, FCG, fCg; ACK, aCk; we have

$$\left.\begin{array}{l} FG : fg :: CF : Cf \\ CG : Cg :: CF : Cf \\ AK : ak :: CA : Ca \\ CK : Ck :: CA : Ca \end{array}\right\}$$ i.e. FG, CG, AK, &c. are to fg, Cg, ak, &c. in the ratio of the radius of the circle $AFBE$ to that of the circle $afbe$.

36. To convert sines, tangents, secants, &c. calculated to the radius (r), into others belonging to a circle whose radius is (R), we have only therefore to increase or diminish the *former* in the ratio of $R : r$. If, for instance, it was required to convert the sines, cosines, tangents, secants, &c. which (in the preceding section) were calculated to radius (1), into others belonging to a circle whose radius is 10000, we have only to multiply each of those numbers by 10000.

Thus,

Radius = 1	Radius = 10000
Sine 45° = .7071068	Sine 45° = 7071.068
Cosine 30° = .8660254	Cosine 30° = 8660.254
Tang. 60° = 1.7320508	Tang. 60° = 17320.508
Secant 30° = 1.1547005	Secant 30° = 11547.005
&c.	&c.

CHAP. II.

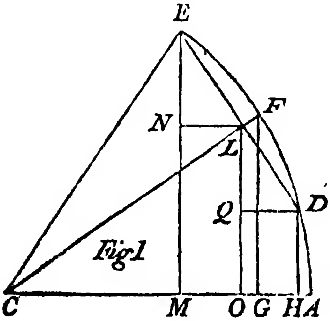
ON THE INVESTIGATION OF TRIGONOMETRICAL FORMULÆ.

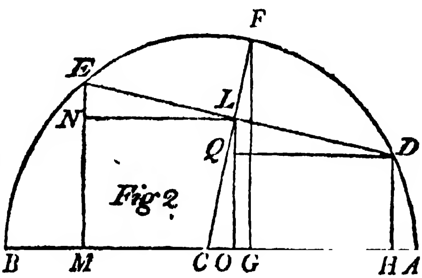
TRIGONOMETRICAL Formulæ are generated by processes purely *algebraical;* but it will be proper to investigate *geometrically* the fundamental Theorem upon which they are built.

VII.

On the method of finding geometrically the sine and cosine of the sum and difference of any two arcs.

37. Let AF, FE, be the two given arcs, of which AF is the *greater;* take $FD=FE$, and draw the chord ED,





which will be bisected by the radius CF in the point L; let fall the *perpendiculars* DH, FG, LO, EM, upon the diameter, and draw DQ, LN, *parallel* to it, meeting LO and EM in the points Q and N. Then $FG=\text{sin.}\ AF$, $CG=\text{cos.}\ AF$, $EL=\text{sin.}\ EF$, $CL=\text{cos.}\ EF$.

Since the arc EF = the arc FD, EL must be equal to LD; and since LN is parallel to DQ, the $\angle ELN$ is equal to the $\angle LDQ$; hence the right-angled triangles ELN, LDQ, are both *equal* and *similar*; $\therefore EN = LQ$, and $NL = QD$. In the parallelograms $MNLO$, $OQDH$, we have $NM = LO$, and $DH = QO$; also $QD = OH$, and $NL = MO$; hence QD, OH, OM, NL, are all equal to each other.

Now the arc $AE = AF + FE$ = *sum* of the arcs,

arc $AD = AF - FD(FE)$ = *difference* of the arcs.

And $EM = \sin.AE$ = sine of the *sum*,

$DH = \sin.AD$ = sine of the *difference*,

$CM = \cos.AE$ = cosine of the *sum*,

$CH = \cos AD$ = cosine of the *difference*.

Again, since FG is parallel to LO, and LN parallel to CO, the triangles CFG, CLO, ENL, are similar;

$$\text{Hence } CF : FG :: CL : LO = \frac{FG \times CL}{CF} = \frac{\sin.AF \times \cos.EF}{\text{rad.}}$$

$$CF : CG :: EL : NE = \frac{CG \times EL}{CF} = \frac{\cos.AF \times \sin.EF}{\text{rad.}}$$

$$CF : CG :: CL : CO = \frac{CG \times CL}{CF} = \frac{\cos.AF \times \cos.EF}{\text{rad.}}$$

$$CF : FG :: EL : NL = \frac{FG \times EL}{CF} = \frac{\sin.AF \times \sin.EF}{\text{rad.}}$$

Now $EM = MN + NE = LO + NE$, or sin of *sum* $= \dfrac{\sin AF \times \cos.EF + \cos AF \times \sin EF}{\text{rad.}}$

Hence $QO = LO - LQ = LO - NE$, or sin. of *dif* $= \dfrac{\sin.AF \times \cos.EF - \cos AF \times \sin EF}{\text{rad}}$

(a) $CM = CO - MO = CO - NL$, or cos. of *sum* $= \dfrac{\cos.AF \times \cos.EF - \sin AF \times \sin EF}{\text{rad.}}$

$CH = CO + OH = CO + NL$, or cos of *dif.* $= \dfrac{\cos AF \times \cos.EF + \sin.AF \times \sin EF}{\text{rad}}$

(a) In Fig. 2, where AE is *greater than* 90°, we have $CM = MO - CO$, $\therefore -CM = CO - MO$, in this case the *cosine* is *negative*, which will be explained in Art 67.

VIII.

On the Formulæ derived immediately from the foregoing Theorem.

Previous to the investigation of these Algebraic Formulæ, it will be necessary to exhibit the system of notation by which the operations are conducted.

39. Let a and b be any two arcs, of which a is the *greater*; then

The *sine* of a is expressed by	$\sin. a$	The sine of their *sum* is expressed by	$\sin\ (a+$
cosine	$\cos. a$	. . . . . *difference* .	$\sin.\ (a-$
tangent	$\tan. a$	. . . . . *half* their sum .	$\sin. \frac{1}{2}(a+$
cotangent	$\text{cotan}. a$	. . . . *half* their differen	$\sin\ \frac{1}{2}(a-$
Square of sine . . .	$\sin^2 a$	The *tangent* of their *sum* .	$\tan.\ (a+$
Cube	$\sin^3 a$	. . . . *difference*	$\tan.\ (a-$
Square of tangent . .	$\tan.^2 a$	. . . . half their *sum* .	$\tan\ \frac{1}{2}(a+$
Cube	$\tan.^3 a$	. . . . *difference*,	$\tan. \frac{1}{2}(a-$
&c. &c. &c.		&c. &c. &c. &c.	

40. Now let rad. $=1$, $AF=a$, $EF=b$; then the general expressions for the sine and cosine of the sum and difference of any two arcs as they stand in Art. 38, may be exhibited in the following manner:

$$\sin. (a+b) = \sin. a \times \cos. b + \cos. a \times \sin. b \ (C).$$
$$\sin. (a-b) = \sin. a \times \cos. b - \cos. a \times \sin. b \ (D).$$
$$\cos. (a+b) = \cos. a \times \cos. b - \sin. a \times \sin. b \ (E).$$
$$\cos. (a-b) = \cos. a \times \cos. b + \sin. a \times \sin. b \ (F).$$

The formulæ immediately deducible from these expressions may be divided into *three classes*.

CLASS I.

CLASS I.

This class consists of formulæ derived from them by *addition* and *subtraction*.

Formula 1.

41. *Add* (*D*) to (*C*), then

$\sin.(a+b)+\sin.(a-b)=2\sin.a\times\cos.b,$

or $\sin.a\times\cos.b=\frac{1}{2}\sin.(a+b)+\frac{1}{2}\sin.(a-b).$

Formula 2.

42. *Subtract* (*D*) from (*C*), then

$\sin.(a+b)-\sin.(a-b)=2\cos.a\times\sin.b,$

or $\cos.a\times\sin.b=\frac{1}{2}\sin.(a+b)-\frac{1}{2}\sin.(a-b).$

Formula 3.

43. *Add* (*E*) to (*F*), we have

$\cos.(a+b)+\cos.(a-b)=2\cos.a\times\cos.b;$

$\therefore\ \cos.a\times\cos.b=\frac{1}{2}\cos.(a+b)+\frac{1}{2}\cos.(a-b).$

Formula 4.

44. *Subtract* (*E*) from (*F*), then

$\cos.(a-b)-\cos.(a+b)=2\sin.a\times\sin.b,$

or $\sin.a\times\sin.b=\frac{1}{2}\cos.(a-b)-\frac{1}{2}\cos.(a+b).$

CLASS II.

In the *second* Class are placed such formulæ as may be immediately derived from those in Class I. by making $a+b=p$, and $a-b=q$; in which case $a=\frac{1}{2}(p+q)$, and $b=\frac{1}{2}(p-q)$; then, from

Formula 1. $\sin.p+\sin.q=2\sin.\frac{1}{2}(p+q)\cos.\frac{1}{2}(p-q).$

. . . . . 2. $\sin.p-\sin.q=2\cos.\frac{1}{2}(p+q)\sin.\frac{1}{2}(p-q).$

. . . . . 3. $\cos.p+\cos.q=2\cos.\frac{1}{2}(p+q)\cos.\frac{1}{2}(p-q).$

. . . . . 4. $\cos.q-\cos.p=2\sin.\frac{1}{2}(p+q)\sin.\frac{1}{2}(p-q).$

But

But it is evident that it is not necessary to consider p and q as the sum and difference of a and b, any longer than whilst the substitution is actually making. When this substitution is once made, the expressions containing p and q become true for any arcs whatever; to preserve therefore an *uniformity of notation*, we shall put a and b for p and q in these latter expressions, and we then have

Formula 5.

45. $\sin. a+\sin. b=2 \sin. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b)$.

Formula 6.

46. $\sin. a-\sin. b=2 \cos. \tfrac{1}{2}(a+b) \sin. \tfrac{1}{2}(a-b)$.

Formula 7.

47. $\cos. a+\cos. b=2 \cos. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b)$.

Formula 8.

48. $\cos. b-\cos. a=2 \sin. \tfrac{1}{2}(a+b) \sin. \tfrac{1}{2}(a-b)$.

CLASS III.

By Art. 15, if rad. $=1$, $\tan.=\dfrac{\sin.}{\cos.}$; and by Art. 16, cotan. $=\dfrac{1}{\tan.}=\dfrac{\cos.}{\sin.}$; and in this third Class are placed the formulæ which arise from *dividing* those of Class II. by each other in succession, and substituting tan. for $\dfrac{\sin.}{\cos.}$, cotan. for $\dfrac{\cos.}{\sin.}$, tan. for $\dfrac{1}{\text{cotan.}}$, or cotan. for $\dfrac{1}{\tan.}$.

Formula 9.

49. $$\frac{\sin. a+\sin. b}{\sin. a-\sin. b}=\frac{\sin. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b) \sin. \frac{1}{2}(a-b)}=\frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)}.$$

Formula 10.

50. $$\frac{\sin.a+\sin.b}{\cos.a+\cos.b}=\frac{\sin.\frac{1}{2}(a+b)\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)\cos.\frac{1}{2}(a-b)}=\frac{\sin.\frac{1}{2}(a+b)}{\cos.\frac{1}{2}(a+b)}=\tan.\tfrac{1}{2}(a+b).$$

Formula 11.

51. $$\frac{\sin.a+\sin.b}{\cos.b-\cos.a}=\frac{\sin.\frac{1}{2}(a+b)\cos.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)\sin.\frac{1}{2}(a-b)}=\frac{\cos.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a-b)}=\text{cotan.}\tfrac{1}{2}(a-b).$$

Formula 12.

52. $$\frac{\sin.a-\sin.b}{\cos.a+\cos.b}=\frac{\cos.\frac{1}{2}(a+b)\sin.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)\cos.\frac{1}{2}(a-b)}=\frac{\sin.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a-b)}=\tan.\tfrac{1}{2}(a-b).$$

Formula 13.

53. $$\frac{\sin.a-\sin.b}{\cos.b-\cos.a}=\frac{\cos.\frac{1}{2}(a+b)\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)\sin.\frac{1}{2}(a-b)}=\frac{\cos.\frac{1}{2}(a+b)}{\sin.\frac{1}{2}(a+b)}=\text{cotan.}\tfrac{1}{2}(a+b).$$

Formula 14.

54. $$\frac{\cos.a+\cos.b}{\cos.b-\cos.a}=\frac{\cos.\frac{1}{2}(a+b)\cos.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)\sin.\frac{1}{2}(a-b)}=\frac{\text{cotan.}\frac{1}{2}(a+b)}{\tan.\frac{1}{2}(a-b)}$$

To this class may be added *three* other formulæ, which arise from making $b=0$ in formulæ 10, 11, 12, 13, or 14; in which case, *sin.* $b=0$, and *cos.* b (=radius) $=1$.

Formula 15.

55. Make $b=0$, in formula 10, or 12; then,

$$\frac{\sin.a}{1+\cos.a}=\tan.\tfrac{1}{2}a=\frac{1}{\text{cotan.}\frac{1}{2}a}.$$

Formula 16.

56. Make $b=0$, in formula 11, or 13; then,

$$\frac{\sin.a}{1-\cos.a}=\text{cotan.}\tfrac{1}{2}a=\frac{1}{\tan.\frac{1}{2}a}.$$

Formula

Formula 17.

57. Make $b=0$, in formula 14; then,

$$\frac{1+\cos. a}{1-\cos. a}=\frac{\text{cotan.}\,\frac{1}{2}a}{\tan.\,\frac{1}{2}a}=\text{cotan.}^2\,\tfrac{1}{2}a, \text{ or } \frac{1}{\tan.^2\,\frac{1}{2}a}.$$

18.

58. *Invert* the expression in formula 17; then

$$\frac{1-\cos. a}{1+\cos. a}=\tan.^2\tfrac{1}{2}a.$$

IX.

On the investigation of Formulæ for finding the sine and cosine of multiple arcs.

59. In Formula 1st, (Art. 41.) transpose sin. $(a-b)$ to the other side of the equation; then,

$$\sin. (a+b)=2\cos. b\times\sin. a-\sin. (a-b).$$

For a in this equation, substitute b, $2b$, $3b$, $4b$, &c. successively; and we have,

$$\begin{aligned}
\sin. 2b&=2\cos. b\times\sin. b.\\
\sin. 3b&=2\cos. b\times\sin. 2b-\sin. b.\\
\sin. 4b&=2\cos. b\times\sin. 3b-\sin. 2b.\\
\sin. 5b&=2\cos. b\times\sin. 4b-\sin. 3b.\\
\&c.&=\&c.\\
\sin. nb&=2\cos. b\times\sin. (n-1)b-\sin. (n-2b.
\end{aligned}$$

60. In Formula 3d, (Art. 43.) transpose cos. $(a-b)$ to the other side of the equation; then,

$$\cos. (a+b)=2\cos. b\times\cos. a-\cos. (a-b).$$

For a in this equation, substitute b, $2b$, $3b$, $4b$, &c. successively; and we have,

$\cos. 2b = 2\cos^2. b - 1,$*
$\cos. 3b = 2\cos. b \times \cos. 2b - \cos. b,$
$\cos. 4b = 2\cos. b \times \cos. 3b - \cos. 2b,$
$\cos. 5b = 2\cos. b \times \cos. 4b - \cos. 3b,$
&c. = &c.
$\cos. nb = 2\cos. b \times \cos. (n-1)\, b - \cos. (n-2)\, b.$

From which it appears, that if the sine and cosine of any arc b be given, the sines and cosines of the multiple arcs $2b$, $3b$, $4b$, $5b$, &c., nb may be found in succession.

X.

On the investigation of Formulæ for finding the tangent and cotangent of multiple arcs.

To do this, we must find the tangents of the sum and difference of any two arcs a and b.

61. Now by Art. 15, when rad. $=1$, tan. $=\frac{\sin.}{\cos.}$, hence

$$\tan. (a+b) = \frac{\sin. (a+b)}{\cos. (a+b)} = (\text{by Art. } 40) \frac{\sin. a \times \cos. b + \cos. a \times \sin. b}{\cos. a \times \cos. b - \sin. a \times \sin. b} =$$

(by dividing numerator and denominator by $\cos. a \times \cos. b$)

$$\frac{\frac{\sin. a}{\cos. a} + \frac{\sin. b}{\cos. b}}{1 - \frac{\sin. a \times \sin. b}{\cos. a \times \cos. b}} = \frac{\tan. a + \tan. b}{1 - \tan. a \times \tan. b}.$$

62. For the same reason, $\tan. (a-b) = \frac{\sin. (a-b)}{\cos. (a-b)} =$

$$\frac{\sin. a \times \cos. b - \cos. a \times \sin. b}{\cos. a \times \cos. b + \sin. a \times \sin. b} = \frac{\frac{\sin. a}{\cos. a} - \frac{\sin. b}{\cos. b}}{1 + \frac{\sin. a \times \sin. b}{\cos. a \times \cos. b}} = \frac{\tan. a - \tan. b}{1 + \tan. a \times \tan. b}.$$

63. Now

* For $\cos. (a-b) = \cos. (b-b) = \cos. 0 = \text{rad.} = 1.$

63. Now in Art. 61, let $b=a$, then

$$\tan.\,2a=\frac{2\tan.\,a}{1-\tan.^2 a}.$$

Let $b=2a$, and we have

$$\tan.\,3a=\frac{\tan.\,a+\tan.\,2a}{1-\tan.\,a\times\tan.\,2a}=\frac{\tan.\,a+\frac{2\tan.\,a}{1-\tan.\,a^2}}{1-\frac{2\tan.^2 a}{1-\tan.^2 a}}$$

$$=\frac{\tan.\,a-\tan.^3 a+2\tan.\,a}{1-\tan.^2 a-2\tan.^2 a}=\frac{3\tan.\,a-\tan.^3 a}{1-3\tan.^2 a}.$$

And thus by substituting for b, in Art. 61, a, $2a$, $3a$, $4a$, &c. successively, we obtain formulæ for $\tan.\,2a$, $\tan.\,3a$, $\tan.\,4a$, $\tan.\,5a$, &c. &c.

64. Since (when rad. $=1$), cotan. $=\frac{1}{\tan.}$, we have

$$\text{cotan.}\,2a=\frac{1}{\tan.\,2a}=\frac{1-\tan.^2 a}{2\tan.\,a}=\frac{1}{2\tan.\,a}-\tfrac{1}{2}\tan.\,a.$$

$$=\tfrac{1}{2}\,\text{cotan.}\,a-\tfrac{1}{2}\tan.\,a.$$

And,

$$\text{cotan.}\,3a=\frac{1}{\tan.\,3a}=\frac{1-3\tan.^2 a}{3\tan.\,a-\tan.^3 a}.$$

&c. $=$ &c.

XI.

On the investigation of Formulæ for expressing the powers *of the sine and cosine of an arc.*

65. By Formula 4th, (Art. 44.) we have

$$\sin.\,a\times\sin.\,b=\tfrac{1}{2}\cos.\,(a-b\tfrac{1}{2}-)\cos.\,(a+b).$$

Let $b=a$, then

$\sin.^2 a=\frac{1}{2}-\frac{1}{2}\cos.\,2a$, and multiplying by $\sin.\,a$,

$\sin.^3 a = \frac{1}{2} \sin. a - \frac{1}{2} \cos. 2a \times \sin. a$

$\quad = \frac{1}{2} \sin. a - \frac{1}{4} \sin. 3a + \frac{1}{4} \sin. a$ *

$\quad = \frac{3}{4} \sin. a - \frac{1}{4} \sin. 3a$——multiply by $\sin. a$, then

$\sin.^4 a = \frac{3}{4} \sin.^2 a - \frac{1}{4} \sin. 3a \times \sin. a$, and substituting for [$\sin.^2 a$ its value just found,

$\quad = \frac{3}{8} - \frac{3}{8} \cos. 2a - \frac{1}{4} \sin. 3a \times \sin. a$

$\quad = \frac{3}{8} - \frac{3}{8} \cos. 2a - \frac{1}{8} \cos. 2a + \frac{1}{8} \cos. 4a$ †

$\quad = \frac{3}{8} - \frac{1}{2} \cos. 2a + \frac{1}{8} \cos. 4a.$

&c. = &c.

By proceeding in this manner, we obtain expressions for any *powers* of the sine, in terms of the sine and cosine of the arc or its multiples.

66. By Formula 3d, (Art. 43,) we have,

$$\cos. a \times \cos. b = \tfrac{1}{2} \cos. (a+b) + \tfrac{1}{2} \cos. (a-b).$$

Let $b = a$, then

$\cos.^2 a = \frac{1}{2} \cos. 2a + \frac{1}{2}$, or $\frac{1}{2} + \frac{1}{2} \cos. 2a$; mult. by $\cos. a$, then

$\cos.^3 a = \frac{1}{2} \cos. a + \frac{1}{2} \cos. 2a \times \cos. a$

$\quad = \frac{1}{2} \cos. a + \frac{1}{4} \cos. 3a + \frac{1}{4} \cos. a$ ‡

$\quad = \frac{3}{4} \cos. a + \frac{1}{4} \cos. 3a$; multiply by $\cos. a$, then

cos.

---

* By Formula 2d, (Art. 42,) $\cos. a \times \sin. b = \frac{1}{2} \sin. (a+b) - \frac{1}{2} \sin. (a-b)$; for a put $2a$, and for b put a, then $\cos. 2a \times \sin. a = \frac{1}{2} \sin. 3a - \frac{1}{2} \sin. a$, $\therefore \frac{1}{2} \cos. 2a \times \sin. a = \frac{1}{4} \sin. 3a - \frac{1}{4} \sin. a$.

† By Formula 4th, (Art. 44,) $\sin. a \times \sin. b = \frac{1}{2} \cos. (a-b) - \frac{1}{2} \cos. (a+b)$, for a put $3a$, and for b put a, then $\sin. 3a \times \sin. a = \frac{1}{2} \cos. 2a - \frac{1}{2} \cos. 4a$, $\therefore \frac{1}{4} \sin. 3a \times \sin. a = \frac{1}{8} \cos. 2a - \frac{1}{8} \cos. 4a$

‡ By Formula 3d, (Art. 43,) $\cos. a \times \cos. b = \frac{1}{2} \cos. (a+b) + \frac{1}{2} \cos. (a-b)$, for a put $2a$, and for b put a, then $\cos. 2a \times \cos a = \frac{1}{2} \cos. 3a + \frac{1}{2} \cos. a$, $\therefore \frac{1}{2} \cos. 2a \times \cos. a = \frac{1}{4} \cos. 3a + \frac{1}{4} \cos. a$.

$\cos.^4 a = \frac{3}{4}\cos.^2 a + \frac{1}{4}\cos.\,3a \times \cos.\,a$; and substituting for [$\cos.^2 a$ its value just found,

$$= \tfrac{3}{8} + \tfrac{3}{8}\cos.\,2a + \tfrac{1}{4}\cos.\,3a \times \cos.\,a,$$

$$= \tfrac{3}{8} + \tfrac{3}{8}\cos.\,2a + \tfrac{1}{8}\cos.\,4a + \tfrac{1}{8}\cos.\,2a, \|$$

$$= \tfrac{3}{8} + \tfrac{1}{2}\cos.\,2a + \tfrac{1}{8}\cos.\,4a.$$

&c. = &c.

And thus we obtain expressions for any *powers* of the cosine, in terms of the cosine of the arc or its multiples.

XII.

On the variation of the sine, cosine, versed sine, tangent, and secant, through the four quadrants of the circle.

Previous to tracing the variation of these lines round the circle, it is necessary to observe, that geometrical quantities are measured from some given point or line, and, when expressed *algebraically*, are reckoned + or −, according as they lie on the *same* or *opposite* sides of that point or line.

67. Thus, in the circle *ADBE*, if the *sines* of the arcs in the semicircle *ADB* are reckoned +, the *sines* of the arcs in the semicircle *BEA* (lying on the *opposite* side

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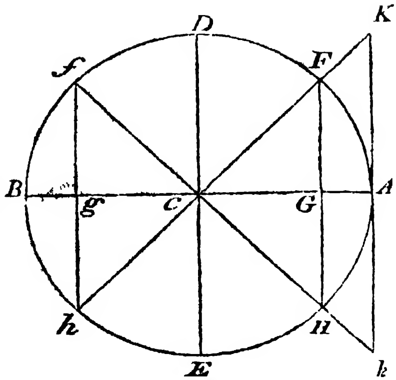
‖ In formula of Note (*), for a put $3a$, and for b put a, then $\cos\,3a \times \cos.\,a = \frac{1}{2}\cos\,4a + \frac{1}{2}\cos.\,2a$, $\therefore \frac{1}{4}\cos.\,3a \times \cos.\,a = \frac{1}{8}\cos\,4a + \frac{1}{8}\cos.\,2a$.



side of the diameter AB), will be reckoned —; and if the *cosines* of the arcs in the first quadrant AD be reckoned +, the *cosines* of arcs in the second quadrant DB (lying on the *opposite* side of the center C), must be reckoned — . Since $\tan. = \frac{\sin.}{\cos.}$, the *tangents* of these arcs will be *positive* or *negative*, according as the sine and cosine have the *same* *different* signs; and since $\sec. = \frac{1}{\cos.}$, the *secants* of those arcs will be *positive* or *negative*, according as the cosine is *positive* or *negative*. With respect to the *versed sines*, since *they* are measured from A, they will be altogether *positive*; in the semicircle ADB they will vary from 0 to *diameter*; and in the semicircle BEA they will vary from *diameter* to 0.

With this explanation, the following Table, exhibiting the *variation of the sine, cosine, tangent, and secant, through the four quadrants of the circle,* will be readily understood.

In first quadrant AD.

The *Sine* increases from 0 to radius, and is +.
Cosine decreases from radius to 0, and is +.
Tangent increases from 0 to infinity, and is +.
Secant increases from radius to infinity, and is +.

In second quadrant DB.

The *Sine* decreases from radius to 0, and is +.
Cosine increases from 0 to radius, and is −.
Tangent decreases from infinity to 0, and is −.
Secant decreases from infinity to radius, and is −.

In third quadrant BE.

The *Sine* increases from 0 to radius, and is −.
Cosine decreases from radius to 0, and is −.
Tangent increases from 0 to infinity, and is +.
Secant increases from radius to infinity, and is +.

In fourth quadrant EA.

The *Sine* decreases from radius to 0, and is −.
Cosine increases from 0 to radius, and is +.
Tangent decreases from infinity to 0, and is −.
Secant decreases from infinity to radius, and is −.

Take any arc AF, and make $Df = DF$; (*See Figure opposite*) draw the chords FH, fh, perpendicular to the diameter AB; join CF, Cf, Ch, CH, and produce them to meet the tangent at A in the points K, k. Then, from the *definitions* of *sine*, *cosine*, *tangent*, and *secant*, it appears that

FG is

FG is the *sine* of the arc AF
fg . . . . of the arc Af
gh . . . . of the arc ABh
GH. . . . . of the arc ABH

From the construction of the Figure, it is easily proved that $FG=fg=gh=GH$.

CG is the *cosine* of the arc AF, and of the arc ABH
Cg . . . . of the arc Af, and of the arc ABh } & $CG=Cg$

AK is the *tangent* of the arc AF, and of the arc ABh
Ak . . . . of the arc Af, and of the arc ABH } & $AK=Ak$.

CK is the *secant* of the arc AF, and of the arc ABh
Ck . . . . of the arc Af, and of the arc ABH } & $CK=Ck$

Now let the arc $AF=a$, and a *semicircular* arc or arc of $180°=\pi$; then, since arc $AF=fB=Bh=AH$, we have,

$$\begin{aligned}
\text{Arc } Af &= \pi-fB = \pi-AF=\pi-a.\\
ABh &= \pi+Bh = \pi+AF=\pi+a.\\
ABH &= 2\pi-AH=2\pi-AF=2\pi-a.
\end{aligned}$$

Hence,			
$FG=\sin.a$	$fg=\sin\overline{\pi-a}$	$gh=\sin.\overline{\pi+a}$	$GH=\sin.\overline{2\pi-a}$
$CG=\cos a$	$Cg=\cos.\overline{\pi-a}$	$Cg=\cos\overline{\pi+a}$	$CG=\cos\overline{2\pi-a}$
$AK=\tan a$	$Ak=\tan.\overline{\pi-a}$	$AK=\tan\overline{\pi+a}$	$Ak=\tan\overline{2\pi-a}$
$CK=\sec a$	$Ck=\sec.\overline{\pi-a}$	$CK=\sec.\overline{\pi+a}$	$Ck=\sec\overline{2\pi-a}$

But when these lines are expressed *algebraically*, $fg=+FG$, gh and $GH=-FG$, $Cg=-CG$; $Ak=-AK$; and $Ck=-CK$; from which we deduce,

$\sin.\overline{\pi-a}=\sin a$*	$\cos.\overline{\pi-a}=-\cos.a$	$\tan.\overline{\pi-A}=-\tan.a$	$\sec.\overline{\pi-a}=-\sec a$
$\sin.\overline{\pi+a}=-\sin.a$	$\cos.\overline{\pi+a}=-\cos.a$	$\tan.\overline{\pi+A}=+\tan.a$	$\sec.\overline{\pi+a}=+\sec.$
$\sin\overline{2\pi-a}=-\sin a$	$\cos.\overline{2\pi-a}=+\cos.a$	$\tan.\overline{2\pi-A}=-\tan a$	$\sec.\overline{2\pi-a}=-\sec.a$

For

* Since $\pi-a$ is the *supplement* of the arc a, it appears that the *sine* of the *supplement* of any angle is the *same* with the

For a more general exhibition of a table of this kind, and for many very important Trigonometrical Theorems applicable to purposes purely algebraical, the Reader is referred to Mr. WOODHOUSE's Treatise on Plane and Spherical Trigonometry.

the *sine* of that angle; and that the *cosine*, *tangent*, and *secant* of the supplement of any angle is the same as the *cosine*, *tangent*, and *secant* of that angle, but with a *negative* sign.

CHAP.

CHAP. III.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

FROM the Formulæ exhibiting the value of the sine, cosine, tangent, &c. in Sect. II. it appears, that if the *sine* of an arc be known, the rest may be immediately found; and by means of the formulæ investigated in Sect. IX. if the sine and cosine of any arc be given, we can find the sine and cosine of any *multiple* of that arc. Hence then it is evident, that if the sine and cosine of *one degree, minute, second*, &c. be known arithmetically, we could calculate the arithmetical value of the sine, cosine, tangent, &c. of *every degree, minute, second*, &c. of the quadrant. We shall therefore begin with shewing the method of finding the sine and cosine of an arc of 1′.

XIII.

Method of finding the sine and cosine of an arc of 1′.

68. The semiperiphery of a circle whose radius is 1, is 3.141592653; and since it is divided into 180°, and each degree into 60 minutes, the number of *minutes* contained in it is 180×60, or 10800; the length of an *arc of* 1′ therefore is $\frac{3.141592653}{10800}$, or .000290888.

Let

Let a be any arc of a circle whose radius is 1, then

$$\text{* sin. } a = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \&c.$$

$$\therefore\ a - \text{sin. } a = \frac{a^3}{2.3} - \frac{a^5}{2.3.4.5} + \&c.$$

Hence $arc\ 1' - sin.\ 1' = \frac{\overline{.000290888}|^3}{2.3} - \frac{\overline{.000290888}|^5}{2.3.4.5} =$ .0000000000041 ; from which it appears, that the difference between an *arc of* 1′ and its *sine* is so small as not to affect their respective values for the first *ten places of decimals*; and as Tables calculated for *seven places of decimals* are sufficiently exact for all common purposes, the *arc* and *sine* may in this case be considered as *equal to each other*; i. e. sin. 1′=.000290888 to radius 1; and therefore cos. $1' = (\sqrt{1 - \overline{\text{sin. } 1'}|^2} =) \sqrt{1 - \overline{.000290888}|^2} = \sqrt{1 - .0000000846158544} = \sqrt{.999.999.915.384171456}$ $= 99999996$ very nearly.

XIV.

Method of constructing a Table of sines, cosines, tangents, &c. for every degree and minute of the quadrant, to seven places of decimals.

Since cos. 1′=.99999996, 2 cos. 1′ must be equal to 1.99999992; call this quantity m. The *nearest* value of .000290888 to *seven* places of decimals is .0002909. Now let b, in the series at the end of Art. 59, be *an arc of* 1′; for sin.

* For the investigation of this series, the Reader is referred to Vince's Fluxions, Prop 103.

sin. b, and 2 cos. b, substitute .0002909 and m respectively; and we have

69. For the *sines*.

sin. $2'=2\cos.1\times\sin.1'=m\times.0002909=.0005818\ (a)$.
sin. $3=2\cos.1'\times\sin.2'-\sin.1'=m\times a-.0002909=.0008727(b)$
sin. $4'=2\cos.1'\times\sin.3'-\sin.2'=m\,b-a \qquad =.0011636(c)$.
sin. $5'=2\cos.1'\times\sin.4'-\sin.3'=m\,c-b \qquad =.0014544$.
&c. = &c. &c.

70. For the *cosines*.

cos. $2'=2\cos 1'\times\cos.1'-1=m\times.99999996-1 \qquad =.9999998(d)$
cos $3'=2\cos.1'\times\cos.2'-\cos.1'=m\times d-.99999996=.9999996(e)$
cos. $4'=2\cos 1'\times\cos.3'-\cos 2'=m\times e-d \qquad =.9999993$
&c. = &c &c.

In this manner we proceed to find the sines and cosines of every degree and minute of the quadrant, as far as 30°; the whole difficulty of the operation consisting only in the multiplication of each successive result by the quantity (m) From 30° to 60° the sines may be found by mere *subtraction*. To shew the method of doing this, it is necessary to have recourse to Formula 1. where we have

$$\sin\overline{a+l}+\sin.\overline{a-b}=2\sin.a\cos.b.$$

$$\left.\begin{array}{l}\text{Let } a=30^\circ,\\ \text{then } \sin.a=\tfrac{1}{2};\end{array}\right\}\begin{array}{l}\therefore \sin.\overline{30^\circ+b}+\sin.\overline{30^\circ-b}=2\times\tfrac{1}{2}\times\cos b=\cos.l,\\ \text{or } \sin.\overline{30^\circ+b}=\cos.b-\sin\overline{30^\circ-b}\end{array}$$

Let $b=1', 2', 3', 4$, &c. then
sin. $30^\circ\,1'=\cos.1-\sin.29^\circ\,59$.
sin. $30^\circ\,2'=\cos.2-\sin.29^\circ\,58'$.
sin. $30^\circ\,3'=\cos.3'-\sin.29^\circ\,57'$.
&c. = &c. − &c.

Which

Which being continued to 60°, the *cosines* also will be known to 60°; for

$$\cos. 30^\circ\ 1' = \sin. 59^\circ\ 59'.$$
$$\cos. 30^\circ\ 2' = \sin. 59^\circ\ 58'.$$
$$\cos. 30^\circ\ 3' = \sin. 59^\circ\ 57'.$$
$$\&c. = \&c.$$

The sines and cosines from 60° to 90° are known from the sines and cosines between 0° and 30°; thus,

sin. 60° 1′=cos. 29° 59′.	cos. 60° 1′=sin. 29° 59′.
sin. 60° 2′=cos. 29° 58′.	cos. 60° 2′=sin. 29° 58′.
sin. 60° 3′=cos. 29° 57′.	cos. 60° 3′=sin. 29° 57′.
&c. = &c.	&c. = &c.

71. For the *versed sines*.

Having found the *sines* and *cosines*, the *versed sines* are found by *subtracting* the cosines from radius in arcs *less* than 90°, and by *adding* the cosines to radius in arcs *greater* than 90°.

Thus, $\text{ver. sin. } 1' = 1 - \cos. 1' = .00000004.$

$$\text{ver. sin. } 2' = 1 - \cos. 2' = .0000002.$$
$$\text{ver. sin. } 3' = 1 - \cos. 3' = .0000004.$$
$$\text{ver. sin. } 4' = 1 - \cos. 4' = .0000007.$$
$$\&c. = \&c.$$

$$\text{ver. sin. } 90^\circ\ 1' = 1 + \cos. 1' = 1.00000004.$$
$$\text{ver. sin. } 90^\circ\ 2' = 1 + \cos. 2' = 1.0000002.$$
$$\text{ver. sin. } 90^\circ\ 3' = 1 + \cos. 3' = 1.0000004.$$
$$\&c. = \&c.$$

72. For the *tangents* and *cotangents*.

When radius = 1, $\tan. a = \frac{\sin. a}{\cos. a}$; hence,

tan.

$$\tan. 1' = \frac{\sin. 1'}{\cos. 1'} = \text{cotan}. 89° 59'.$$

$$\tan. 2' = \frac{\sin. 2'}{\cos. 2'} = \text{cotan}. 89° 58'.$$

$$\tan. 3' = \frac{\sin. 3'}{\cos. 3'} = \text{cotan}. 89° 57'.$$

$$\&c. = \&c. = \&c.$$

In this manner it will be necessary to proceed till we arrive at tan. 45°, after which the tangents (and consequently the *cotangents*) may be found by a more simple method. For by Art. 61, 62.

$$\tan. \overline{a \pm b} = \frac{\tan. a + \tan. b}{1 \mp \tan. a \times \tan. b}.$$

$$\left.\begin{array}{r}\text{Let } a = 45°,\\ \text{then} \tan. a = 1;\end{array}\right\} \therefore \tan. \overline{45° + b} = \frac{1 + \tan. b}{1 - \tan. b},$$

$$\text{and } \tan. \overline{45° - b} = \frac{1 - \tan. b}{1 + \tan. b}.$$

$$\text{Hence } \tan. \overline{45° + b} - \tan. \overline{45° - b} = \frac{1 + \tan. b}{1 - \tan. b} - \frac{1 - \tan. b}{1 + \tan. b}.$$

$$= \frac{\overline{1 + \tan. b}|^2 - \overline{1 - \tan. b}|^2}{\overline{1 - \tan. b}|^2}$$

$$= \frac{4 \tan. b}{\overline{1 - \tan. b}|^2}.$$

$$\text{But by Art. 63. } \tan. 2b = \frac{2 \tan. b}{\overline{1 - \tan. b}|^2};$$

$$\therefore 2 \tan. 2b = \frac{4 \tan. b}{\overline{1 - \tan. b}|^2}.$$

$$\text{Hence } \tan. \overline{45° + b} - \tan. \overline{45° - b} = 2 \tan. 2b,$$

$$\text{or } \tan. \overline{45° + b} = \tan. \overline{45° - b} + 2 \tan. 2b.$$

Let

Let $b = 1', 2', 3', 4'$, &c. then

$$\tan. 45° 1' = \tan. 44° 59' + 2 \tan. 2' = \text{cotan}. 44° 59'.$$
$$\tan. 45° 2' = \tan. 44° 58' + 2 \tan. 4' = \text{cotan}. 44° 58'.$$
$$\tan. 45° 3' = \tan. 44° 57' + 2 \tan. 6' = \text{cotan}. 44° 57'.$$
$$\&c. \quad = \quad \&c.$$

By this means we obtain the tangents and cotangents for every degree and minute of the quadrant.

73. For the *secants* and *cosecants*.

The secants and cosecants of the *even* minutes of the quadrant may be found from Art. 24, where we have,

$$\text{Tan}. a + \sec. a = \text{cotan. of } \tfrac{1}{2} \text{comp}. a;$$
$$\therefore \sec. a = \text{cotan}. \tfrac{1}{2} \text{comp}. a - \tan. a.$$

Let $a = 2', 4', 6', 8'$, &c.

$$\text{then } \sec. 2 = \text{cotan}. 44° 59' - \tan. 2' = \text{cosec}. 89° 58$$
$$\sec. 4' = \text{cotan}. 44° 58' - \tan. 4' = \text{cosec}. 89° 56'.$$
$$\sec. 6 = \text{cotan}. 44° 57' - \tan. 6' = \text{cosec}. 89° 54'.$$
$$\&c. \quad = \quad \&c.$$

where the secants (and consequently the *cosecants*) are known from the tangents and contangents being known.

With respect to the *odd* minutes of the quadrant, we must have recourse to the expression $\sec. a = \frac{1}{\cos. a}$.

Let $a = 1', 3', 5', 7'$, &c. then

$$\sec. 1' = \frac{1}{\cos. 1'} = \text{cosec}. 89° 59'.$$

$$\sec. 3 = \frac{1}{\cos. 3'} = \text{cosec}. 89° 57'.$$

$$\sec. 5' = \frac{1}{\cos. 5'} = \text{cosec}. 89° 55'.$$

$$\&c. \quad = \quad \&c.$$

By

By means therefore of these formulæ the secants and cosecants for the whole quadrant are known.

XV.

On the investigation of formulæ of verification.

We have thus shewn the method of constructing the Trigonometrical Canon of signs, cosines, tangents, &c. for every degree and minute of the quadrant; the *mode of arranging them* in Tables must be learned from the Tables themselves, and the explanations which accompany them. We shall now shew the method of investigating certain formulæ, which, from their utility in rectifying any errors which may be made in these laborious arithmetical calculations, are called *Formulæ of verification.*

In Sect. V. we gave the method of finding the sines, cosines, tangents, &c. of a variety of arcs from the established properties of arcs of 45° and 30°; the values of the sines, cosines, &c. deduced by this independent method, would serve as a very proper check to the computist in the process of calculation, and in that respect the formulæ from which they were derived may be considered as *formulæ of verification.* But from the principles laid down in the preceding chapter, a *vast variety* of formulæ of this kind might be deduced. We shall select only *one*, which may serve as a specimen of the rest.

74. In the isosceles triangle, described in the 10th Prop. of the Fourth Book of Euclid (see Figure in that book), since each of the angles at the base is *double* of the angle at the vertex, it is evident that $5\,BAD=180°$, or $BAD=36°$; the base BD therefore is the *chord* of an

arc

arc of 36°, and consequently *twice the sine of* 18°; $\therefore \frac{1}{2}BD = \sin.\ 18°$.

$$\left.\begin{aligned} \text{Let } BD &= x, \\ AB &= 1; \\ \text{then } BC &= AB - AC, \\ &= AB - BD, \\ &= 1 - x. \end{aligned}\right\} \begin{aligned} &\text{Since } AB \times BC = BD^2, \\ &\text{we have } 1 \times \overline{1-x} = x^2; \\ &\therefore x^2 + x = 1, \\ &\text{and } x^2 + x + \tfrac{1}{4} = 1 + \tfrac{1}{4} = \tfrac{5}{4}, \end{aligned}$$

$$\text{or } x + \tfrac{1}{2} = \frac{\sqrt{5}}{2};$$

$$\therefore x = \frac{\sqrt{5}-1}{2},$$

$$\text{and } \tfrac{1}{2}x = \frac{\sqrt{5}-1}{4} = \sin.\ 18°.$$

Hence $\overline{\cos.\ 18°}|^2 = (1 - \overline{\sin.\ 18°}|^2)\ 1 - \frac{6-2\sqrt{5}}{16} = \frac{5+\sqrt{5}}{8}$.

By Art. 40, $\cos\ \overline{a+b} = \cos\ a \times \cos.\ b - \sin.\ a \times \sin.\ b$.

Let $b = a$, then $\cos.\ 2a = \overline{\cos.\ a}|^2 - \overline{\sin.\ a}|^2$;

$$\begin{aligned} \therefore \cos.\ 36° &= \overline{\cos.\ 18°}|^2 - \overline{\sin.\ 18°}|^2 \\ &= \frac{5+\sqrt{5}}{8} - \frac{6-2\sqrt{5}}{16} \\ &= \frac{10+2\sqrt{5}-6+2\sqrt{5}}{16} \\ &= \frac{4\sqrt{5}+4}{16} = \frac{\sqrt{5}+1}{4} = \sin.\ 54°. \end{aligned}$$

By Formula 1,

If $a = 54°$

$\sin\ \overline{54°+b} + \sin\ \overline{54°-b} = 2 \sin\ 54° \times \cos.\ b = \frac{\sqrt{5}+1}{2} \times \cos.\ b\ (X)$

If $a = 18°$,

$\sin\ \overline{18°+b} + \sin\ \overline{18°-b} = 2 \sin\ 18° \times \cos.\ b = \frac{\sqrt{5}-1}{2} \times \cos\ b\ (Y)$

Subtract

Subtract Y from X; then we have

$$\sin.\overline{54^\circ+b}+\sin.\overline{54^\circ-b}-\sin.\overline{18^\circ+b}-\sin.\overline{18^\circ-b}=\cos. b,$$

where different values may be substituted for b, at the pleasure of the computist.

Let

$b=10^\circ$, then $\sin. 64^\circ+\sin. 44^\circ-\sin. 28^\circ-\sin. 8^\circ=\cos. 10^\circ$
$b=15^\circ$, . . . $\sin. 69^\circ+\sin. 39^\circ-\sin. 33^\circ-\sin. 3^\circ=\cos. 15^\circ$
&c. &c. &c. &c.

Example.

In Sherwin's Tables (5th Edition), where the natural sines, cosines, tangents, &c. are computed to radius 10000, it appears that

sin. 64° =	8987.940	sin. 28° =	4694.714
sin. 44° =	6946.584	sin. 8° =	1391.731
	15934.524		6086.445
	6086.445		
	9848.079 = cos. 10° according to the formula.		

Now, in the *same* Tables, the cosine of 10° is calculated at 9848.078, from which it appears, that there is some inaccuracy in the *last* figure of the numbers expressing the value either of sin. 64°, sin. 44°, sin. 28°, cos. 10°, or sin. 8°.

Again,

sin. 69° =	9335.804	sin. 33° =	5446.390
sin. 39° =	6293.204	sin. 3° =	523.360
	15629.008		5969.750
	5969.750		
	9659.258 = cos. 15° according to the formula.		

In the same Tables, the cos. 15° stands at 9659.258; from which we may conclude, that sin. 69°, sin. 39°, sin. 33°, cos. 15°, and sin. 3°, are rightly computed.

XVI.

On the construction of tables of logarithmic sines, cosines, tangents, &c.

75. We have already shewn the method of calculating arithmetically a table of sines, cosines, tangents, &c. for every degree and minute of the quadrant; which, thus expressed in *parts of the radius*, are called *natural* sines, cosines, &c.* But to facilitate the actual solution of problems in Plane and Spherical Trigonometry, it is necessary that we be furnished with the *logarithms* of these quantities.(*) To do this would be only to find the logarithms of the numbers as they stand in the tables, pages 34, 35; but as those tables are calculated for radius (1), the sines and cosines are all *proper fractions*; their logarithms, therefore, would all be *negative*. To avoid this, the common tables of logarithmic sines, cosines, &c. are calculated to a radius of 10^{10} or 10000000000, in which case log. radius $= 10 \times$ log. $10 =$ (for log. $10 = 1$) $10 \times 1 = 10.0000000$.

Now, let $s =$ sine of any arc to radius (1);
then, by Art. 36, $10^{10} \times s =$ equal sine of the same arc to radius 10^{10}.
But log. $10^{10} \times s = 10 \times$ log. $10 +$ log. $s = 10 +$ log. s.

Hence, to find the logarithm of the sine of any arc to the radius 10^{10}, we have only to add 10 to the logarithm of that sine when calculated to the radius (1).

EXAMPLE.

(*) For the method of calculating Logarithmic Tables, and for a full explanation of the nature and use of Logarithms, the reader is referred to the last chapter of the "*Elements of Algebra*"

EXAMPLE 1. To find the logarithmic sine of 1′.

By Sect. XIII. sine of 1′ to radius (1) $= .0002909 = \frac{2909}{10000000} = s$,

$\therefore \log. s = \log. 2909 - \log. 10000000 = 3.4637437 - 7 = \bar{4}.4637437$.

Hence, $10 + \log. s = 10 + \bar{4}.4637437 = 6.4637437 = \log.$ sine of 1′.

Ex. 2. To find the logarithmic sine of 4° 15′.

Natural sine of $4^{\circ}\ 15' = .0074108 = \frac{74108}{1000000} = s$;

$\therefore \log. s = \log. 74108 - \log. 1000000 = 4.8698651 - 6. = \bar{2}.8698651$.

Hence, $10 + \log. s = 10 + \bar{2}.8698651 = 8.8698651 = \log. \sin. 4^{\circ}\ 15'$.

And in this manner the logarithmic *cosines* may be found.

76. Having found the logarithmic *sines* and *cosines*, the logarithmic *tangents*, *secants*, *cotangents*, and *cosecants*, are found (from the expressions in Sect. II.) merely by addition and subtraction, in the following manner:

Tan. $= \frac{\text{rad.} \times \sin.}{\cos.}$, $\therefore \log. \tan. = \log. \text{rad.} + \log. \sin. - \log. \cos. = 10 + \log. \sin. - \log. \cos.$

Sec. $= \frac{\overline{\text{rad.}}|^2}{\cos.}$, $\therefore \log. \sec. = 2 \log. \text{rad.} - \log. \cos. \quad . \quad . = 20 - \log. \cos.$

Cotan. $= \frac{\overline{\text{rad.}}|^2}{\tan.}$, $\therefore \log. \text{cotan.} = 2 \log. \text{rad.} - \log. \tan. \quad . \quad . = 20 - \log. \tan.$

Cosec. $= \frac{\overline{\text{rad.}}|^2}{\sin.}$, $\therefore \log. \text{cosec.} = 2 \log. \text{rad.} - \log. \sin. \quad . \quad . = 20 - \log. \sin.$

77. To find the logarithmic *versed sines*.

By Art. 20,

$$\text{ver. sin.} = \frac{\overline{\text{chord}}|^2}{\text{diam.}} = \frac{\overline{2 \sin. \frac{1}{2} \text{arc}}|^2}{2 \text{ rad.}} = \frac{2 \times \overline{\sin. \frac{1}{2} \text{arc}}|^2}{\text{rad.}};$$

$\therefore \log. \text{ver. sin.} = \log. 2 + 2 \log. \sin. \frac{1}{2} \text{arc} - \log. \text{rad.}$

EXAMPLE.

EXAMPLE. To find log. versed sine of 30°.

Log. ver. sin. of $30° = \log. 2 + 2 \log. \sin. 15° - \log. \text{rad.}$

$$\begin{aligned} \text{Now } \log. 2 &= .3010300, \\ 2 \log. \sin. 15° &= 18.8259924 \\ \hline &\ 19.1270224 \\ \text{Log. rad.} &= 10.0000000 \\ \hline \therefore\ & \ \ 9.1270224 = \log. \text{ver. sin. of } 30°. \end{aligned}$$

We have thus shewn the method of constructing tables of *natural* and *logarithmic* sines, cosines, versed sines, tangents, co-tangents, secants, and co-secants. But the actual calculation of these tables, or any part of them, is not the object of a tract of this kind.

CHAP. IV.

ON THE

METHOD OF ASCERTAINING THE RELATION BETWEEN THE

SIDES AND ANGLES OF PLANE TRIANGLES;

AND ON THE

MEASUREMENT OF HEIGHTS AND DISTANCES.

BEFORE we proceed to apply the principles laid down in the three preceding Chapters to ascertain the relation which obtains between the sides and angles of plane triangles, and to the actual measurement of the heights and distances of objects, it will be necessary to investigate a few general Rules or Theorems of the following nature.

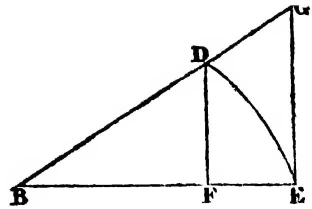
XVII.

On the investigation of Theorems for ascertaining the relation which obtains between the sides and angles of right-angled and oblique-angled triangles.

78. In the right-angled triangle DBF, if the hypothenuse BD be made radius, the sides DF, BF become respectively the *sine* and *cosine* of the angle adjacent to the base.

With

With BD as radius, describe the circular arc DE, and produce the base BF to E; then, by Art^s. 7, 11, DF is the *sine*, and BF is the *cosine* of the angle DBF, to the radius BD.



79. In the right-angled triangle BEG, if the side BE be made radius, the other side EG becomes the *tangent*, and the hypothenuse BG becomes the *secant* of the angle adjacent to the base.

With BE as radius, describe circular arc ED cutting the hypothenuse BG in the point D; then EG touches the arc ED, and, by Art. 9, EG becomes the *tangent* and BG becomes the *secant* of the angle GBE, to the radius BE.

80. In any plane triangle, the *sides* are to each other as the *sines of the angles opposite to them.*

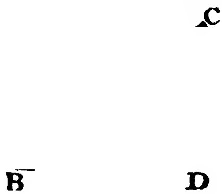


Fig. 1.

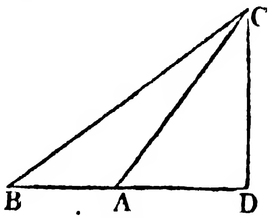


Fig. 2.

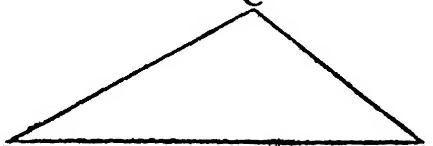
In the oblique-angled triangle ABC, let fall the perpendicular CD upon the base, or upon the base *produced*; then, by Art. 77,

The side BC : the side CD :: radius : sine of the angle CBD,
and side CD : the side CA :: sine of angle CAD : radius;
$\therefore$ ex æquo,
The side BC : the side CA : the sine of $\angle CAD$: the sine of $\angle CBD$,
:: sin. $\angle$ oppos. to BC : sin. $\angle$ oppos. to CA.

In

In the figure where the perpendicular CD falls upon the base BA *produced*, the angle CAB is the supplement of the angle CAD; but by Art. 67, the sine of the *supplement* of any angle is the same with the sine of the angle itself; in this case therefore the sine of CAB might be substituted for the sine of CAD, and the proposition becomes general for any plane triangle.

81. In any plane triangle ABC, the *sum* of the sides BC, CA : their difference :: the *tangent* of *half the sum* of the angles CBA, BAC at the base : the *tangent* of *half their difference*.



Let BC be the *longer* side, and let the angle $CBA=b$, $BAC=a$.

Now by Art. 80, $BC : CA :: \sin. a : \sin. b$;

$\therefore BC+CA : BC-CA :: \sin. a+\sin. b. : \sin. a-\sin. b.$

$$\text{Hence } \frac{BC+CA}{BC-CA}=\frac{\sin. a+\sin. b}{\sin. a-\sin. b}.$$

$$\left.\begin{array}{r}\text{But by}\\ \text{Formula 49,}\end{array}\right\} \frac{\sin. a+\sin. b}{\sin. a-\sin. b}=\frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)};$$

$$\therefore \frac{BC+CA}{BC-CA}=\frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)};$$

or

or $BC+CA : BC-CA :: \tan. \frac{1}{2}(a+b) : \tan. \frac{1}{2}(a-b)$.*

82. Referring to the Figures in Art. 80, we have

In Fig. 1, by Euc. B. II. Prop. 13, $BC^2=AB^2+AC^2-2AB\times AD$,

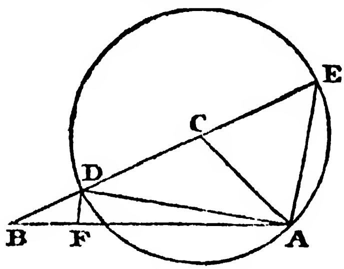
$$\therefore AD=\frac{AB^2+AC^2+BC^2}{2AB}.$$

In Fig. 2, by Euc. B. II, Prop. 12, $BC^2=AB^2+AC^2+2AB\times AD$,

$$\therefore -AD=\frac{AB^2+BC^2-BC^2}{2AB}$$

* This proposition may be demonstrated *geometrically*, thus:

Let ABC be any triangle whose *shorter* side is AC; with centre C, and radius CA, describe the circle ADE, and produce BC to E; join EA, AD, and draw DF at right angles to AD.



Now $BE=BC+CE=BC+CA=$ the *sum* of the sides, and $BD=BC-CD=BC-CA=$ the *difference* of the sides; the *exterior* angle $ACE=BAC+CBA=a+b$, and this is the angle at the *centre*; hence the angle ADC (which is the angle at the *circumference*) $=\frac{1}{2}ACE=\frac{1}{2}(a+b)$; but the angle CAD is equal to the angle ADC, $\therefore CAD=\frac{1}{2}(a+b)$, and the angle $BAD=BAC-CAD=a-\frac{1}{2}(a+b)=\frac{1}{2}(a-b)$

Let DA be made radius, then, by Art. 79, since the angle DAE in a semicircle is a right angle, AE is the tangent of the angle ADC, or $AE=\tan. \frac{1}{2}(a+b)$, ; and DF is the tangent of [illegible] to the same radius, or $DF=\tan. \frac{1}{2}(a-b)$. Again, since AE, DF are each perpendicular to DA, they are *parallel*, and consequently by sim. triangles we have,

$$BE : BD :: AE : DF$$
$$\text{or } BC+CA : BC-CA :: \tan. \tfrac{1}{2}(a+b) : \tan. \tfrac{1}{2}(a-b)$$

In

In each of these Figures; if AC be made radius, we have $AC : AD ::$ rad. : cos. of the angle CAD, $\therefore$ cos. $CAD = \frac{\text{rad.} \times AD}{AC}$, and $-$cos. $CAD = \frac{-\text{rad.} \times AD}{AC}$.

Let the three *angles* at the points A, B, C be called a, b, c respectively; and the three *sides* (BC, CA, BA) opposite to them be called A, B, C respectively; then $AD = \frac{B^2 + C^2 - A^2}{2C}$ in the first Figure, and $-AD = \frac{B^2 + C^2 - A^2}{2C}$ in the second Figure. Substitute these values for AD and $-AD$ in the foregoing expressions, then we have

In Fig. 1. $\quad$ cos. $CAD = \left(\frac{\text{rad.} \times AD}{AC} = \right) \frac{\text{rad.} (B^2 + C^2 - A^2)}{2\,B.C}$.

In Fig. 2. $-$cos. $CAD = \left(\frac{-\text{rad.} \times \cdot}{AC} = \right) \frac{\text{rad.} (B^2 + C^2 - A^2)}{2\,B.C}$.

Now in Fig. 2, the angle CAD is the *supplement* of the angle CAB, $\therefore$ (by Art. 67.) $-$cos. CAD is the cosine of the angle CAB (or a). Hence, in general,

$$\text{cos. } a := \frac{\text{rad. } (B^2 + C^2 - A^2)}{2\,B.C}.$$

This expression may be transformed into another more convenient for logarithmic calculation, by the following process:

By Art. 14, ver. sin. $a = $ rad. $-$ cos. a,

$$= \text{rad.} - \frac{\text{rad. } (B^2 + C^2 - A^2)}{2\,B.C},$$

$= \frac{1}{2}$ rad.

$$=\frac{\frac{1}{2}\text{ rad. }(2\,B.C-B^2-C^2+A^2)}{B.C},$$

By Art. 34, $\sin.^2\frac{1}{2}a=\frac{1}{2}\text{ rad.}\times\text{ver. sin. }a,$

$$=\frac{\frac{1}{4}\text{ rad.}^2(2\,B.C-B^2-C^2+A^2)}{B.C},$$

$$=\frac{\frac{1}{4}\text{ rad.}^2\,(A^2-(B-C)^2)}{B.C},$$

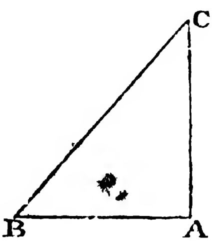
$$=\frac{\frac{1}{4}\text{ rad}^2\,(\overline{A+B-C}\times\overline{A-B+C})}{B.C}.$$

Hence $\sin.\frac{1}{2}a=\frac{\frac{1}{2}\text{ rad.}\sqrt{\overline{A+B-C}\times\overline{A-B+C}}}{\sqrt{B.C}},$

and $\log.\sin.\frac{1}{2}a=\log.\frac{1}{2}\text{ rad.}+\frac{1}{2}\log.(A+B-C)+\frac{1}{2}\log.(A-B+C)-\frac{1}{2}\log.B-\frac{1}{2}\log.C.$

XVIII.

On the application of the foregoing Theorems to finding the relation between the sides and angles of right-angled triangles.



83. Given the hypothenuse BC, and side AC; to find side AB, and $\angle^s$ B, C.

By Eucl. 47.1. $BC^2=AB^2+AC^2$;

$\therefore\ AB^2=BC^2-AC^2,$

and $AB=\sqrt{BC^2-AC^2}.$

By Art. 77, $BC : AC :: \text{rad.} : \sin.B=\frac{\text{rad.}\times AC}{BC}.$

Lastly, $\angle C=90^\circ-\angle B.$

 EXAM.

Example.

Let $BC=56,$ } Then $AB=\sqrt{56^2.-36^2}=\sqrt{1840}=42.89.$
$AC=36.$ } $\sin.\ \angle B=\dfrac{\text{rad.}\times AC}{BC}=\dfrac{\text{rad}\times 36}{56};$

$\therefore$ log. sin. $\angle B=$ log. rad. + log. 36 − log. 56.

Now log. rad. = 10.0000000

log. 36 = 1.5563025

11.5563025

log. 56 = 1.74 [illegible]

log. sin. $\angle B=$ 9.808 [illegible] $\angle B$ = 40° 1′.

And $\angle C=90^\circ-\angle B=90^\circ-40^\circ\,1'=49^\circ\,59'$

81. Given side $AB,$ } to find the hypothenuse $BC,$ and
and side $AC,$ } $\angle^s B, C.$

By Euclid, 47. 1. $BC=\sqrt{AB^2+AC^2}.$

By Art. 79, $AB : AC ::$ rad. : tan. $\angle B=\dfrac{\text{rad.}\times AC}{AB}.$

And $\angle C=90^\circ-\angle B.$

Example.

Let $AB=36,$ } Then $BC=\sqrt{36^2+40^2}=53.81,$
$A[illegible]=40.$ }

tan. $\angle B=\dfrac{\text{rad.}\times 40}{36},$

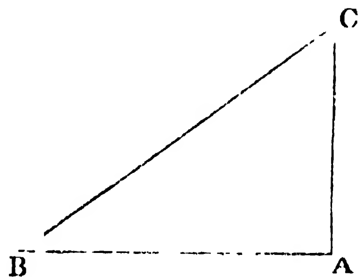
$\therefore$ log. tan. $\angle B=$ log. rad. + log. 40 − log. 36.

Now

$$\begin{aligned}
\text{Now log. rad.} &= 10.0000000 \\
\text{log. } 40 &= 1.6020600 \\
&\overline{11.6020600} \\
\text{log. } 36 &= 1.5563025 \\
\text{log. tan. } \angle B &= \overline{10.0457575}\,; \ \therefore \angle B = 48^\circ\ 1'.
\end{aligned}$$

And $\angle C = 90^\circ - \angle B = 41^\circ\ 59'$.

85. Given the hypothenuse BC, and $\angle B$; to find $\angle C$, and sides AC, AB.



Now $\angle C = 90^\circ - \angle B$.

By Art. 77, $BC : AC :: \text{rad.} : \text{sin.} \angle B$; $\therefore AC = \dfrac{BC \times \text{sin.} \angle B}{\text{rad.}}$

And by Eucl. 47. 1. $AB = \sqrt{BC^2 - AC^2}$.

EXAMPLE.

Let $\left.\begin{aligned} BC &= 100, \\ \angle B &= 49^\circ. \end{aligned}\right\}$ Then $\angle C = 90^\circ - \angle B = 90^\circ - 49^\circ = 41^\circ$.

$$AC = \frac{100 \times \text{sin. } 49^\circ}{\text{rad.}}\,;$$

$\therefore \text{log. } AC = \text{log.} 100 + \text{log. sin.} 49^\circ - \text{log. rad.}$

Now

$$\begin{array}{rr} \text{Now log. } 100 = & 2.0000000 \\ \text{log. sin. } 49^\circ = & 9.8777799 \\ \hline & 11.8777799 \\ \text{log. rad.} = & 10.0000000 \\ \hline \text{log. } AC = & 1.8777799 \end{array} ; \therefore AC = 75.47.$$

$$AB = \sqrt{100^2 - 75.47^2} = 65.607. ^*$$

86. Given side AB, and $\angle B$, } to find the $\angle C$, side AC, and hypothenuse BC.

Now $\angle C = 90^\circ - \angle B$.

By Art. 79, $AB : AC :: \sin. C : \sin. B$; $\therefore AC = \dfrac{AB \times \sin. B}{\sin. C}$.

And $BC = \sqrt{AB^2 + AC^2}$.

EXAMPLE.

Let $AB = 70$, $\angle B = 50^\circ$. } Then $\angle C = 90^\circ - 50^\circ = 40^\circ$,

$$AC = \frac{70 \times \sin. 50^\circ}{\sin. 40^\circ};$$

$\therefore$ log. AC = log. 70 + log. sin. 50° − log. sin. 40°.

Now

* The value of AB might also be found by *Logarithms* in following manner:

$$AB = \sqrt{BC^2 - AC^2} = \sqrt{\overline{BC + AC} \times \overline{BC - AC}};$$

$\therefore$ log. $AB = \frac{1}{2}$ log. $\overline{BC + AC} + \frac{1}{2}$ log. $\overline{BC - AC} = \frac{1}{2}$ log. $175.47 + \frac{1}{2}$ log. 24

$$\begin{array}{rr} \text{Now } \frac{1}{2} \text{ log. } 175.47 = & 1.1221014 \\ \frac{1}{2} \text{ log. } 24.53 = & .6948487 \\ \hline \therefore \text{log. } AB = & 1.8169501 \end{array}, \text{ or } AB = 65.607.$$

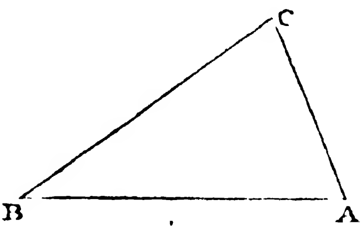
$$\begin{aligned}
\text{Now log. } 70 &= 1.8450980 \\
\text{log. sin. } 50^\circ &= 9.8842540 \\
&\overline{11.7293520} \\
\text{log. sin. } 40^\circ &= 9.8080675 \\
&\overline{} \\
\text{log. } AC &= 1.9212845; \therefore AC = 83.42.
\end{aligned}$$

And $BC = \sqrt{70^2 + 83.42^2} = 108.90.$

XIX.

On the application of the foregoing Theorems to determining the sides and angles of oblique-angled triangles.

87. Given the *two angles* B, A, and the side BC *opposite to one of them;* to find the $\angle C$, and the other sides AB, AC.



Now $\angle C = 180^\circ - (\angle A + \angle B)$.

By Art. 80, $BC : AC :: \sin. \angle A : \sin. \angle B$; $\therefore AC = \dfrac{BC \times \sin. \angle B}{\sin. \angle A}$.

And $BC : AB :: \sin. \angle A : \sin. \angle C$; $\therefore AB = \dfrac{BC \times \sin. \angle C}{\sin. \angle A}$.

EXAM.

Example.

Let $BC=62$, $\angle B=35°$, $\angle A=60°$. Then $\angle C=180°-(\angle A+\angle B)=180°-(60°+35°)=85°$.

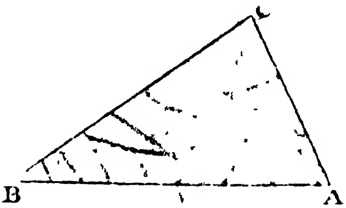
$AC=\dfrac{62\times\sin.35°}{\sin.60°}$; $\therefore$ log. AC = log. 62 + log. sin. 35°

$-$log. sin. 60° = 1.6134521, and $AC=41.06$.

$AB=\dfrac{62\times\sin.85°}{\sin.60°}$; $\therefore$ log. AB = log. 62 + log. sin. 85°

$-$log. sin. 60° = 1 8532053, and $AB=71.31$.

88. Given the *two sides* BC, AC, and $\angle B$ *opposite* to AC; to find the angles A, C, and the other side AB.



By Art. 80, $BC:AC$ :: sin. $\angle A$: sin. $\angle B$; $\therefore$ sin. $\angle A=\dfrac{BC\times\sin.\angle B}{AC}$.

$\angle C=180°-(\angle A+\angle B)$.

And $AC:AB$ :: sin. $\angle B$: sin. $\angle C$; $\therefore$ $AB=\dfrac{AC\times\sin.\angle C}{\sin.\angle B}$.

Exam

EXAMPLE.

Let $BC=50$, $AC=40$, $\angle B=32°$. Then $\sin.\angle A=\frac{50\times\sin.32°}{40}$, and log. sin. $\angle A$ $=\log.50+\log.\sin.32°-\log.40=9.8211197$;

$\therefore \angle A=41°\ 28'$.

$\angle C=180°-\overline{41°\ 28'+32°}=106°\ 32'$.

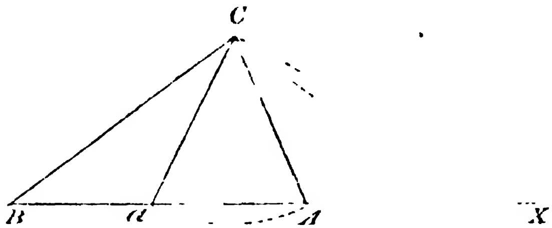
$AB=\frac{40\times\sin.106°\ 32'}{\sin.32°}=\left(\begin{array}{l}\text{for sin. of an } \angle = \text{sin. of } \textit{supplement}, \\ \therefore \sin.106°\ 32'=\sin.73°\ 28'.\end{array}\right)$

$\frac{40\times\sin.73°\ 28'}{\sin.32°}$; $\therefore \log.AB=\log.40+\log.\sin.73°\ 28'$

$-\log.\sin.32°=1.8595123$; hence $AB=72.36$.*

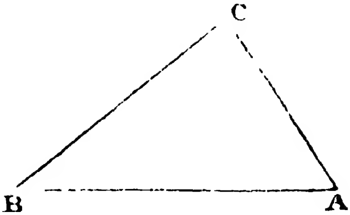
89. Given

* In finding the sine of the $\angle A$ in *this* case, an *ambiguity* arises; for as the sine of the *supplement* of any angle is the *same* with the sine of the angle, the angle thus found may be either A or $180°-A$. But there will be no ambiguity, except in the case when $\angle B$ *is acute*, and BC *greater than the side opposite to the* $\angle B$. For if the $\angle B$ be *obtuse*, then it is evident $\angle A$ must be *acute*. If $\angle B$ be *acute*, and BC *less than the side opposite to the* $\angle B$, then take $Cb=CB$, and draw any other



line CX cutting Bb produced in X, then no line equal to CX can be drawn between B and b, and BCX will be the only triangle which can answer the conditions required; but if BC be

89. Given the *two sides* BC, CA, and the *included angle* C; to find $\angle$s B, A, and side AB.



$\angle$s$(A+B) = 180° - \angle C$; $\therefore \angle A + \angle B$, and consequently $\frac{1}{2}(\angle A + \angle B)$, is known.

By Art. 81, $BC+CA : BC-CA :: \tan.\frac{1}{2}(\angle A+\angle B) : \tan.\frac{1}{2}(\angle A-\angle B)$;

Hence $\tan.\frac{1}{2}(\angle A-\angle B)=\dfrac{(BC-CA)\times\tan.\frac{1}{2}(\angle A+\angle B)}{BC+CA}$;

$\therefore \frac{1}{2}(\angle A - \angle B)$ is known.

By Art. 80, $BC : BA :: \sin.\angle A : \sin.\angle C$; $\therefore AB=\dfrac{BC\times\sin.\angle C}{\sin.\angle A}$.

EXAM.

be *greater than the side opposite to the* $\angle B$, then a circular arc Aa may be described, cutting Bl in A, a, so that there will be *two* triangles, BCA, BCa, in which *two sides, and an* $\angle$ *opposite to one of them*, shall be given quantities.

For instance, let $BC=50$, CA or $Ca=40$, $\angle B=32°$. Then the triangles BCA will be the triangle determined by assuming the $\angle A=41°28'$; but 9.8211197 (see Example) is also the log sin of its *supplement* 138° 32′.

Hence,

$\angle BaC$ (which is the supplement of CaA or CAa) $=138° 32'$; and $\angle BCa = 180° - \overline{138° 32' + 32°} = 9° 28'$; in which case $Ba=\dfrac{40\times\sin\ 9° 28'}{\sin.\ 32°}$; $\therefore \log\ Ba = \log.\ 40 + \log.\ \sin.\ 9° 28' - \log.\ \sin.\ 32° = 1.0939470$, or $Ba=12.415$; $\therefore$ the triangles BCA, BCa, will each of them answer the conditions required.

EXAMPLE I.

$$\left.\begin{array}{r}\text{Let } BC=60,\\ AC=50,\\ \angle C=80°.\end{array}\right\}\begin{array}{l}\text{Then } BC+CA=110, \text{ and } BC-CA=10.\\ \text{And } A+B=180°-\angle C=180-80°=100°;\\ \therefore \tfrac{1}{2}(\angle A+\angle B)=50°.\end{array}$$

Hence $\tan.\tfrac{1}{2}(\angle A-\angle B)=\left(\frac{(BC-CA)\times\tan.\tfrac{1}{2}(\angle A+\angle B)}{BC+CA}\right)$

$\frac{10\times\tan.\ 50°}{110}$; $\therefore$ log. tan. $\tfrac{1}{2}(\angle A-\angle B)=$ log. 10 + log. tan.

$50°-$ log. $110=9.0347938$, or $\tfrac{1}{2}(\angle A-\angle B)=6°\ 11'$

But $\angle A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B)=50°+6°\ 11'=56°\ 11'$;

and $\angle B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B)=50°-6°\ 11'=43°\ 49'$.

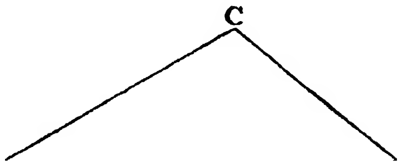
Lastly, $BA=\frac{BC\times\sin.\ \angle C}{\sin.\ \angle A}=\frac{60\times\sin.\ 80°}{\sin.\ 56°\ 11'}$;

$\therefore$ log. $BA=$ log. 60 + log. sin. $80°-$ log. sin. $56°\ 11$

$=1.8519945$, or $BA=71.12$.

90. Given

90. Given the three sides, AB, BC, CA, to find the three angles opposite to them.



For the purpose of applying the expressions in Art. 82, call the three sides BC, CA, AB, A, B, C, and the three angles opposite to them, a, b, c, respectively. Then to determine the angle A (a), we have

$$\cos. a = \frac{\text{rad}\ (B^2+C^2-A^2)}{2\,B.C}.$$

or,

og. sin. $\tfrac{1}{2}a$ = log. $\tfrac{1}{2}$ rad. + $\tfrac{1}{2}$ log. $(A+B-C)$ + log. $(A-B+C)$ − $\tfrac{1}{2}$ log. B − $\tfrac{1}{2}$ log. C, where the *former* or *latter* of these expressions must be used according as the numbers representing the sides are *small* or *large* numbers.

Example I.

$$\left.\begin{aligned}\text{Let } BC&=34,\\ CA&=25,\\ AB&=40,\end{aligned}\right\}\text{then } \cos. a = \frac{\text{rad.}(B^2+C^2-A^2)}{2\,B.C} = \frac{\text{rad.}(40^2+25^2-34}{2\times 40\times 25}$$

$$= \frac{\text{rad.}\ \times 1069}{2000};$$

$$\therefore \log.\cos. a = \log.\text{rad.} + \log. 1069 - \log. 2000$$

$$= 9.7279477,$$

and $a = 57°\ 42'$.

By

By Art. 80, $\sin. b = \dfrac{25 \times \sin. 57^\circ 42'}{34}$,

$\therefore \log. \sin. b = \log. 25 + \log. \sin. 57^\circ 42' - \log. 34$

$= 9.7934524$,

and $b = 38^\circ 25'$.

Lest $c = 180^\circ - (a+b) = 180^\circ - (57^\circ 42' + 38^\circ 25') = 83^\circ 53'$.

Example II.

For the purpose of applying the expression

$\log. \sin. \frac{1}{2} a = \frac{1}{2} \log. \text{rad.} + \frac{1}{2} \log. (A+B-C) + \frac{1}{2} \log. (A-B+C)$
$- \frac{1}{2} \log. B - \frac{1}{2} \log. C$.

$$\left.\begin{aligned} \text{Let } A &= 379.25 \\ B &= 234.15 \\ C &= 415.39 \end{aligned}\right\} \begin{aligned} \text{Then } \log. \tfrac{1}{2} \text{rad.} &= \log. \sin. 30^\circ = 9.6989700 \\ \tfrac{1}{2} \log. (A+B-C) &= \tfrac{1}{2} \log. 198.01 = 1.1483435 \\ \tfrac{1}{2} \log. (A-B+C) &= \tfrac{1}{2} \log. 560.48 = \underline{1.3742839} \end{aligned}$$

$$12.2215974 \ (X).$$

$$\begin{aligned} \tfrac{1}{2} \log. B &= \tfrac{1}{2} \log. 234.15 = 1.1847471 \\ \tfrac{1}{2} \log. C &= \tfrac{1}{2} \log. 415.39 = \underline{1.3092280} \end{aligned}$$

$$2.4939751 \ (Y).$$

Subtract (Y) from (X), then $9.7276223 = \log. \sin. \frac{1}{2}$

Hence $\frac{1}{2} a = 32^\circ 17'$, and $a = 64^\circ 34'$.

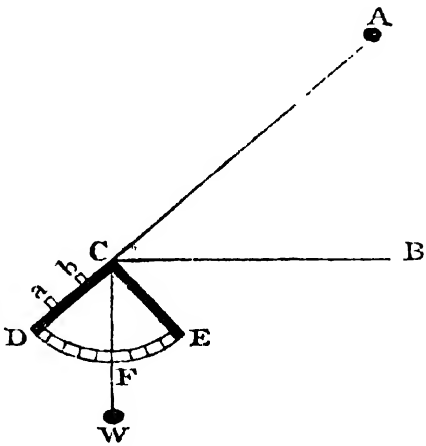
The angles b and c must be found as before.

XX. *On*

XX.

On the Instruments used in measuring Heights and Distances.

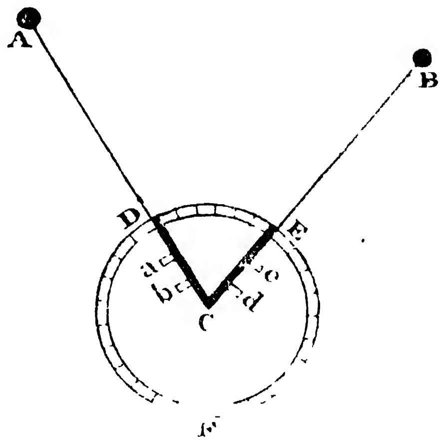
For the mensuration of heights and distances, two instruments (*one* for measuring angles in a *vertical*, and *another* for measuring them in a *horizontal* direction) are required, of which the following is a description.



91. *CDE* is a *graduated* quadrant of a circle, *C* its center, *A* any object, *CB* a line parallel to the horizon, and *CW* a plumb-line hanging freely from *C*, and consequently perpendicular to *CB*. If the quadrant is moved round *C*, till the object *A* is visible through the two sights *a*, *b*, then the arc *EF* will measure the angular distance of the object above the horizon. For the angles *BCW* and *ACE* being right angles, take away the common angle *BCE*, and the remaining angle *ECF* is equal to the remaining angle *ACB*; *EF* therefore (being the measure of the ∠ *ECF*) gives the number of degrees, minutes, &c. of the angle *ACB*. Some such instrument as this must be used for measuring angles in a *vertical* direction.

92. *DCF*

92. *DCF* is a *Theodolite*, or some *graduated* circular instrument, with two indices moveable round the center *C*; *A* and *B* are two objects upon the horizon; when this instrument is so adjusted, that *A* is visible through the sights *a*, *b*, and *B* through the sights *c*, *d*, then the arc *ED* will measure the angular distance (*ACB*) between these two objects.



XXI. *On*

XXI.

On the Mensuration of Heights and Distances.

93. If the object (AE) is *accessible*, as in Fig. 1, let the observer recede from it along ED, till the angle ACB becomes equal to 45°; then, since the angle BAC will in this case be *also* 45°, AB will be equal to BC or ED; measure ED, and to it add BE, the height from which the observation was made, and it will give $AB+BE(AE)$ the *height of the object.*

But if it be not convenient to recede along the line ED till the $\angle ACB$ becomes 45°, let him measure some *given distance* ED, and take with the quadrant the angle ACB; then in the right-angled triangle ACB there is given the *side* BC, and the *angle* ACB, from which the *side* AB may be found, by Art. 84.

EXAMPLE.

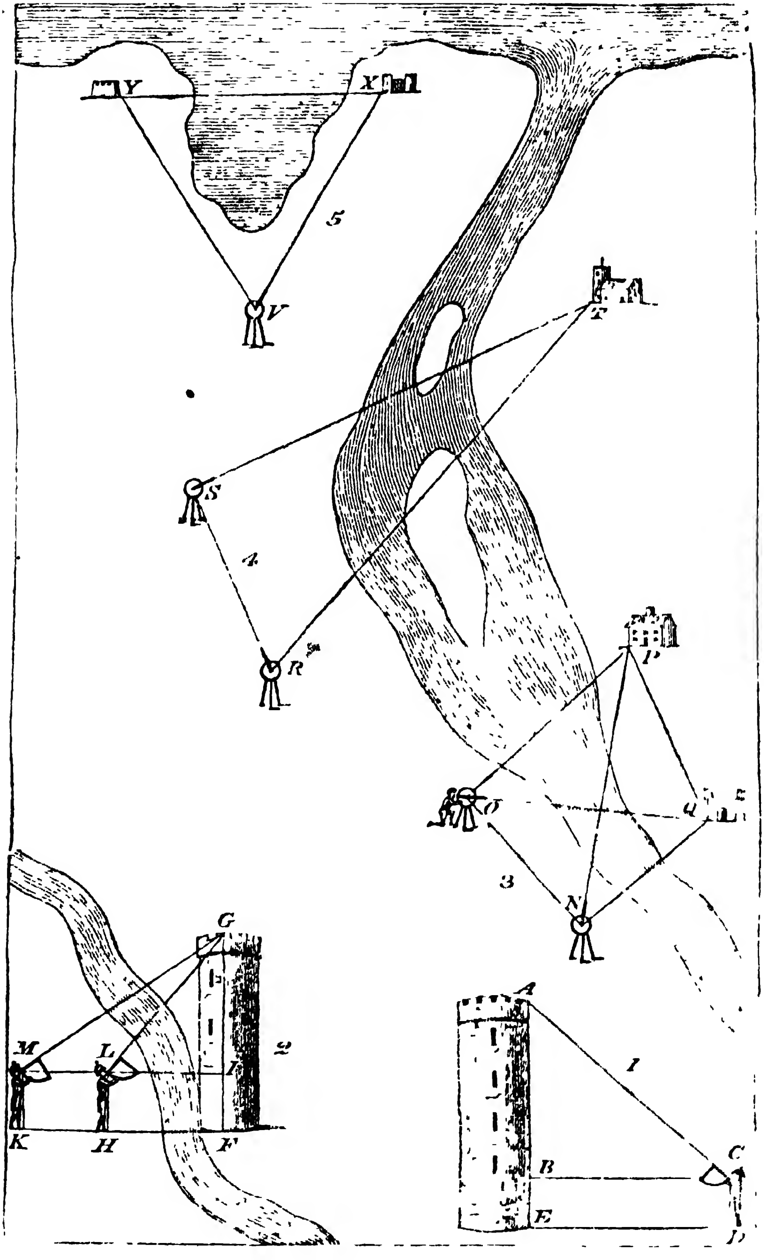
$$\left.\begin{array}{l}\text{Let } BC \text{ or } ED=60 \text{ yards}\\ \angle ABC=47°.\end{array}\right\} \begin{array}{l}\text{Then } BC : AB :: \text{rad.} : \tan. \angle ACB,\\ \text{or } 50 : AB :: R : \tan. 47°;\end{array}$$

$$\therefore AB=\frac{50\times\tan. 47°}{\text{rad.}},$$

and log. $AB=$log. $50+$log. tan. 47°$-$log. rad.$=1.7293141$.

Hence $AB=53.61$ yards; to which if CD or BE be added, it will give AE, *the height of the object.*

94. If the object is *inaccessible*, as GF in Fig. 2.; at some given point H, observe the angle GLI; measure along


Y
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A
1
C
B
E

along some given distance HK, and then observe the angle GMI. In this case, since the *exterior* angle GLI is equal to $GML+MGL$, the angle MGL $(=GLI-GML)$ will be known. In the triangle GML, therefore, we have the *side* ML and *two angles*; from which GL may be determined by Art. 87. Having GL and the angle GLI, the side GI is determined as in Art. 85.

Example.

Let

$$\left.\begin{array}{l} HK \text{ or } LM=100 \text{ yards,} \\ \angle GLI=47^\circ, \\ \angle GMI=36^\circ; \end{array}\right\} \therefore \angle MGL=(GLI-GMI)47^\circ-36^\circ=11^\circ.$$

Now $LM : GL :: \sin.\angle MGL : \sin.\angle GML$,

or $100 : GL :: \sin.11^\circ : \sin.36^\circ$; $\therefore GL=\dfrac{100\times\sin.36^\circ}{\sin.11^\circ}$

Hence $\log.GL=\log.100+\log.\sin.36^\circ-\log.\sin.11^\circ$
$=2.4886199$;
$\therefore GL=308.04$ yards.

Again, $GL : GI :: \text{rad.} : \sin.\angle GLI$,

or $GL : GI :: \text{rad.} : \sin.47^\circ$; $\therefore GI=\dfrac{GL\times\sin.47^\circ}{\text{rad.}}$

Hence $\log.GI=\log.GL+\log\sin.47^\circ-\log.\text{rad.}=2.3527474$;
$\therefore GI=225.29$ yards.

To GI add the height from which the angles were taken, and it will give GF *the height of the object.*

95. By

95. By the following process, a general expression may be investigated for GI, which will apply to all cases of this kind.

$GI : GL :$ sin. L : rad.

$GL : ML ::$ sin. M : sin. MGL (sin. $(L-M)$);

$\therefore GI : ML ::$ sin. $L \times$ sin. M : rad. $\times$ sin$(L-M)$,

and $GI = \dfrac{ML \times \sin L \times \sin M}{\text{rad.} \times \sin (L-M)} = \dfrac{ML \times \sin. L \times \sin. M}{\text{rad.}^3} \times \dfrac{\text{rad}^2}{\sin. (L-M)}$

$= \dfrac{ML \times \sin L \times \sin M \times \text{cosec} (L-M)}{\text{rad.}^3}$, for $\dfrac{\text{rad.}^2}{\sin. (L-M)} = \text{cosec.} (L-M)$ by Art.

Hence log. GI = log ML + log sin L + log. sin. M + log cosec. $(L-M)$ − 3 log rad.

Thus, in the foregoing Example,

log. ML = log. 100	=	2.0000000
log. sin. L = log. sin. 47°	=	9.8641275
log. sin. M = log. sin. 36°	=	9.7692187
log cosec. $(L-M)$ = log. cosec. 11°	=	10.7194012
		32.3527474
3 log. rad.	=	30.0000000
log. GI	=	2.3527474, and GI = 225 29 yards, as before.

96 To find the distance of the object T, (Figure 4.) from the given point S, place at the given point R some small object distinctly visible from S, and then observe the angle TSR; measure the distance SR, and from R observe the angle TRS. In the triangle TSR, we shall then have given SR and the ∠s TSR, TRS; the side ST may therefore be determined by Art. 87.

EXAMPLE.

Let $SR = 150$ yards, $\angle TSR = 91°$, $\angle TRS = 64°$; then $\angle STR = 180° - (91° + 64°) = 25°$

Now $ST : SR :: \sin. \angle TRS : \sin. \angle STR$,

or $ST : 150 :: \sin. 64° : \sin. 25°$; $\therefore ST = \frac{150 \times \sin. 64}{\sin. 25°}$,

Hence Log. $ST = \log. 150 + \log. \sin. 64° - \log. \sin. 25° = 2.5948032$, and $ST = 393.37$ yards.

97. To find the distance between *two* objects, X, Y, *inaccessible* to each other, but *accessible* by the Observer in the directions VX, VY, (Figure 5.); at the given point V, observe the angle XVY, and then measure the line VY. If X is distinctly visible from Y, then the angle XYV may be measured, and the case becomes the *same as the last*, for determining the distance XY. But if X be *not visible* from Y, then both VX and VY must be measured; and having the angle XVY, XY may be found as in Art. 89.

EXAMPLE.

Let $VX = 302$ yards, $VY = 314$ . . . $\angle V = 57° 22'$; then *sum* of $\angle$s$(X+Y) = 180° - 57° 22' = 122° 38'$.

Now

Now

$VY+VX : VY-VX :: \tan.\frac{1}{2}.(X+Y) : \tan.\frac{1}{2}.(X-Y)$,

or $616 : 12 :: \tan. 61°19' : \tan.\frac{1}{2}(X-Y)=\frac{12\times\tan 61°19'}{616}$;

$\therefore \log. \tan.\frac{1}{2}(X-Y)=\log. 12+\log. \tan. 61° 19'-\log. 616$
$=8.5515290$.

Hence $\frac{1}{2}.(X-Y)=2° 2'$; consequently $X=63° 21'$,
and $Y=59° 17'$.

Again,

$XY : YV :: \sin. V : \sin. X$,

or $XY : 314 :: \sin. 57°22' : \sin. 63° 21'$; $\therefore XY=\frac{314\times\sin 57°22'}{\sin. 63°21'}$;

$\therefore \log. XY=\log 314+\log. \sin. 57°22'-\log. \sin. 63°21'$
$=2.4708909$;

and $XY=295.72$ yards.

98. To find the distance PQ between two objects, P and Q, which are *both inaccessible* to the Observer (Fig. 3.); measure a *given distance* ON; from O observe the angles POQ. QON, and from N observe the angles ONP, PNQ; then in the triangle PON will be given the *side* ON and the *two angles* PON, PNO, from which PO may be determined, and in the triangle QON will be given the *side* ON, and the *two angles* QON, ONQ, from which OQ may be found. Having PO, OQ, and the angle POQ, PQ may be determined as in the last case.

EXAMPLE.

Example.

Let $ON = 100$ yards, Hence $\angle PON = 57° + 48° = 105.$
$\angle POQ = 57°$, $\angle QNO = 42° + 49° = 91°.$
$\angle QON = 48°$, $\angle OPN = 180° - (105° + 42°) = 3$
$\angle ONP = 42°$, $\angle OQN = 180° - (91° + 48°) = 4$
$\angle PNQ = 49°.$

Now,

$$QO : ON :: \sin.\ \angle QNO : \sin.\ \angle OQN,$$

$$\text{or } QO : 100 :: \sin.\ 91° \text{ or } 89° : \sin.\ 41°;$$

$$\therefore QO = \frac{100 \times \sin.\ 89°}{\sin.\ 41}.$$

Hence, log $QO =$ log $100 +$ log. sin. $89° -$ log. sin. $41° = 2.1829909$,
and $QO = 152.4$ yards.

Again,

$$PO : ON :: \sin.\ \angle PNO : \sin.\ \angle OPN,$$

$$\text{or } PO : 100 :: \sin.\ 42° : \sin.33°; \therefore PO = \frac{100 \times \sin.\ 42}{\sin.33°}$$

Hence, log $PO =$ log $100 +$ log. sin. $42° -$ log. sin. $33° = 2.0894021$,
and $PO = 122.8$ yards.

Hence, in the triangle POQ, there are given

$$\left.\begin{array}{r} PO = 122.8, \\ OQ = 152.4, \\ \angle POQ = 57°, \end{array}\right\} \text{to find } PQ.$$

$$\angle OPQ + \angle OQP = 180° - POQ = 180° - 57° = 123°;$$

$$\therefore \tfrac{1}{2}(OPQ + OQP) = 61°\ 30'.$$

Now

Now $QO+OP$ $QO-OP$ :: tan. $\frac{1}{2}(OPQ+OQP)$: tan. $\frac{1}{2}(OPQ-OQP)$,

or 275.2 : 29.6 :: tan. 61° 30′ : tan. $\frac{1}{2}(OPQ-OQP)$.

Hence tan. $\frac{1}{2}(OPQ-OQP)=\dfrac{29.6\times\text{tan. }61°30'}{275.2}$

$\therefore$ log. tan. $\frac{1}{2}(OPQ-OQP)=$ log. 29.6 + log. tan. 61° 30′ − log. 275.2
$=9.2968789$,

and $\frac{1}{2}(OPQ-OQP)=11°12'$.

Hence $\angle OPQ=72°42'$, and $\angle OQP=50°18'$.

Lastly,

$QO : PQ ::$ sin. OPQ : sin. POQ,

or $QO : PQ ::$ sin. 72° 42′ : sin. 57°;

$$\therefore PQ=\frac{QO\times\text{sin. }57°}{\text{sin. }72°42'}.$$

Hence log. $PQ=$ log. $QO+$ log. sin. 57° − log. sin. 72° 42′ = 2.1266877

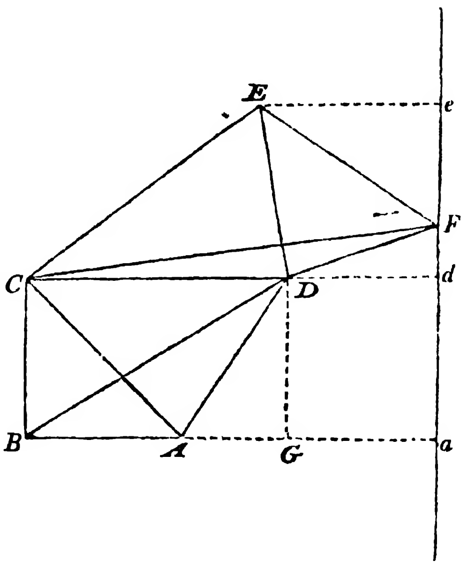
and $PQ=$ 133 87 yards.

XXII. *On*

XXII.

On the manner of constructing a Map of a given surface, and finding its area; with the method of approximating to the area of any given irregular or curve-sided figure.

99. *To construct a map.*—Measure some given distance *AB*; and having selected two objects *C, D,* distinctly visible from *A, B,* observe the angles *CBD, CAD,* as in Art. 98, and find the length of *CD, BC, AD,* in the triangles *ABC, ABD,* by the process made use of in that article. In this manner, the *distance* and *position* of the four points *A, B, C, D,* are determined. In the same manner, by selecting two othe objects *E, F,* distinctly visible from *C, D,* the *distance* and *position* of four other points *C, D, E, F,* may be found. We might thus proceed, by the mensuration of *angles only*, to determine the distance and position of any number of points in a given surface, and to delineate upon paper (by means of a scale) their *relative* position and distance as represented in the figure *ABCEFD*.



100. By

100. By a very easy process we might also determine the length of the part $eFda$ cut off, from a line given in position and passing through any point F, by perpendiculars Ee, Dd, Aa, let fall upon it from the points E, D, A. For the lengths of the lines AD, DF, FE, being found as in Art. 99, and the magnitude of the angles ADG (DG being drawn parallel to da), DFd, EFe being known from the *given position* of the line $EFda$, we have

$$AG : DG \text{ or } da :: \text{rad.} : \cos. ADG, \therefore da = \frac{AG \times \cos. ADG}{\text{rad.}}.$$

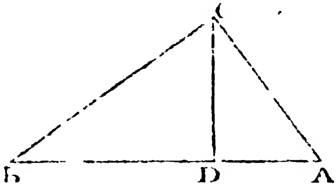
$$DF : Fd \quad :: \text{rad.} : \cos. DFd, \therefore Fd = \frac{DF \times \cos. DFd}{\text{rad.}}.$$

$$FF : EF \quad :: \text{rad.} : \cos. EFe, \therefore EF = \frac{EF \times \cos. EFe}{\text{rad.}}.$$

from which the length of $ad + dF + Fe$ (or $adFe$) is known. If the line passing through F be drawn due north and south, then the length $adFe$ thus determined, is the length of that portion of the meridian which lies between the parallels of latitude passing through the points A, E; and it is upon this principle that the process for measuring the arc of a meridian passing through a given tract of country is conducted.

101. The *area* of the figure $ABCEFD$ is evidently the sum of the areas of all the triangles of which it is composed; we must therefore shew the mode of finding the area of a triangle.

Let ABC be any triangle, and let fall the perpendicular CD upon the base AB; then, since (Eucl. B. I, Prop. 41.)



the area of a triangle is equal to half the area of a parallelogram of the same base and altitude, the area of the triangle ABC is equal to $\frac{1}{2}AB \times CD$. Now BC : CD :: rad. : sin. $\angle B$, $\therefore CD = \frac{BC \times \sin. \angle B}{\text{rad.}}$, and area of triangle ABC $(= \frac{1}{2}AB \times CD) = \frac{\frac{1}{2}AB \times BC \times \sin \angle B}{\text{rad.}} = \frac{AB \times BC \times \sin \angle B}{2 \text{ rad.}}$ hence log. area ABC = log. AB + log. BC + log. sin. $\angle B$ − (log. 2 + log. rad.); for instance, in the triangle ABC of the figure $ABCEFD$, if AB = 100 yards, BC = 90 yards, and $\angle B = 80°$, then

log. AB = log. 100	=	2.0000000
log. BC = log. 90	=	1.9542425
log sin. $\angle B$ = log.sin.80°	=	9.9933515
		13.9475940
* log. 2 + log. rad.	=	10.3010300
log area ABC	=	3.6465640, and area ABC =

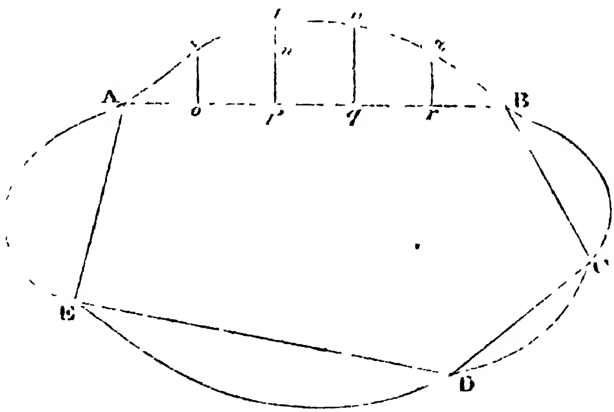
[4431.6 square yards.

And

* Since log 2 + log rad. is in all cases a given quantity, "log area = log base + log. side + log. sin of $\angle$ adjacent to "that side − 10.3010300" is a *general expression* for finding the area of any triangle

And in this manner the areas of the other triangles may be determined; for area of $ACD = \frac{AC \times CD \times \sin. ACD}{2 \text{ rad.}}$, of $DCE = \frac{DC \times CE \times \sin. DCE}{2 \text{ rad.}}$, and of $DEF = \frac{DE \times EF \times \sin. DEF}{2 \text{ rad.}}$.

102. By what has been shewn in the last Article, it appears that the area of any plane-sided Figure may be found by resolving it into its constituent triangles, and then finding the areas of those triangles separately. We are now to explain the method of approximating to the area of an irregular or curved-sided figure (a field for instance), such as is represented in the annexed plate.



After having selected certain points A, B, C, D, E in the perimeter of the Figure, and having made a Map of it and measured the rectilinear figure $ABCDE$ by the method prescribed in Articles 99, 101, a near approximation may be made to the areas of the several curvilinear

parts by means of the following process. Take, for instance, the part cut off by the chord AB. Divide AB into such a number of *equal* parts, Ao, op, pq, qr, rB, that when the perpendiculars os, pt, qv, rx, are drawn from it to the perimeter, the parts As, st, tv, vx, xB may be considered as right lines, without any great deviation from the truth; draw sy parallel to op; and let As, op, &c. each $=m$; then

The triangle $Aos=\frac{1}{2}m\times os$; the figure $sopt=sopy+\Delta syt=m\times py+\frac{1}{2}m\times yt=m(py+\frac{1}{2}yt)$; now $os+pt=2py+yt$, $\therefore \frac{1}{2}(os+pt)=py+\frac{1}{2}yt$: hence the figure $sopt=m\times\frac{1}{2}(os+pt)=\frac{1}{2}m\times os+\frac{1}{2}m\times pt$. For the same reason, $tpqv=\frac{1}{2}m\times pt+\frac{1}{2}m\times qv$; &c. &c. Hence,

$$\begin{array}{llllll}
\Delta Aos & =\frac{1}{2}m\times os & & & & \\
sopt & =\frac{1}{2}m\times os & +\frac{1}{2}m\times pt & & & \\
tpqv & = & \frac{1}{2}m\times pt & +\frac{1}{2}m\times qv & & \\
vqrx & = & & \frac{1}{2}m\times qv & +\frac{1}{2}m\times rx & \\
\Delta rxB & = & & & \frac{1}{2}m\times rx & \\
\hline
\end{array}$$

$\therefore$ area $AtxBrpA=m\times os+m\times pt+m\times qv+m\times rx$ $=(os+pt+qv+rx)m$; i.e. the area of this curvilinear part is nearly approximated to by multiplying the sum of the perpendiculars so, pt, qv, rx, by the length of one of the aliquot parts into which AB is divided. In the same manner we might proceed to measure the curvilinear parts cut off by the chords BC, CD, DE, EA, and thus approximate very nearly to the area of the whole Figure.

XXII. *A*

XXIII.

A few questions for practice in the rules laid down in this Chapter.

103. There is a certain perpendicular rock, from which you can recede only 16 feet, on account of the sea; the angular distance of its highest point, taken at the water's edge by a person 5 feet high, is 80°. QUÆRE, the height of the rock?

ANSWER, 95.74 feet.

104. A person 6 feet high, standing by the side of a river, observed that the top of a tower placed on the *opposite* side, subtended an angle of 59° with a line drawn from his eye parallel to the horizon; receding backwards for 50 feet, he then found that it subtended an angle of only 49°. QUÆRE, the height of the tower, and the breadth of the river?

ANSWER, *Height of tower*= 192.27 feet.
Breadth of river= 111.92 . . .

105. A person walking along a straight terrace AB, 400 feet long, observed, at the end A, the angular distance of an horizontal object C, to be 75° from the terrace; at the end B, the object, viewed in the same manner, formed an angle of 60° only with the terrace. What was the distance of the object C from each end of the terrace?

ANSW. AC=489.89 feet.
BC=546.41 . . . .

106. Two

106. Two objects, A and B, are *visible* and *accessible* from the station C, but are *invisible* and *inaccessible* from each other; the distance AC is 1800 yards, BC 1500 yards, and the $\angle ACB$ is 45°. What is the distance of A from B?

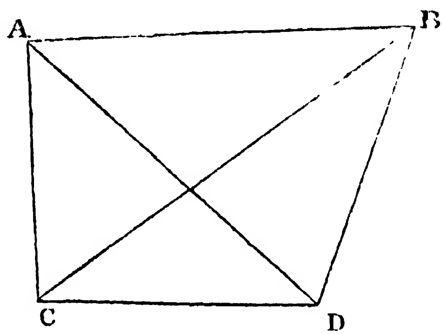
Answ. $AB = 1292.7$ yards.

107. Three objects, A, B, C, are so situated, that $AB = 16$ yards, $BC = 14$ yards, and $AC = 10$ yards. What is the *position* of these objects, with respect to each other?

Answ. $\angle A = 60°$.
$\angle B = 38° \; 12'$.
$\angle C = 81° \; 48'$.

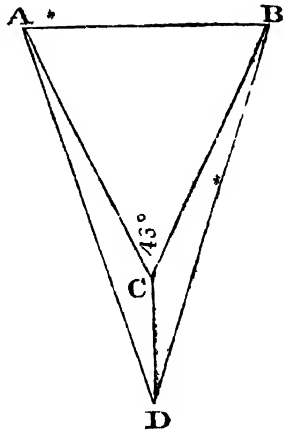
108. To find the distance between the two objects A and B, on supposition that

$CD = 300$ yards.
$\angle ACB = 56°$
$\angle BCD = 37°$
$\angle ADB = 55°$
$\angle ADC = 41°$



Answer, $AB = 341.25$ yards.

109. There



109. There are two objects A, B, so situated, that they are accessible no nearer than C, and *that* in the direction DC, almost perpendicular to the line which joins them.

The $\angle ACB = 46°$,
$ACD = 150°$,
$BCD = 164°$,
$ADC = 20°$,
$CDB = 10°$,
$CD = 100$ yards.

required the distance AB.

ANSW. $AB = 144.67$ yards.

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